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Open sets satisfying the strong meromorphic approximation property

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Abstract. By giving counterexamples we prove that a rationally convex open set D of $\mathbb{C}^n$, where $n \geq 2$, does not satisfy in general the strong meromorphic approximation property in $\mathbb{C}^n$. We also prove that every open set D of a reduced Stein space X of dimension 1 satisfies the strong meromorphic approximation property in X.

1. Introduction

We say that an open set D of a reduced complex space X satisfies the *strong meromorphic approximation property* in X if for every holomorphic function $\varphi \in \mathcal{O}(D)$, for every compact set K of D and for every $\varepsilon > 0$ there exist holomorphic functions $f, g \in \mathcal{O}(X)$ such that $g \not\equiv 0$ on any irreducible component of X, $g \neq 0$ on D and $\|\varphi - (f/g)\|_K < \varepsilon$.

By the theorem of Behnke-Stein [5, Satz 13], which generalizes the rational approximation theorem of Runge [25], every open set D of an open Riemann surface X satisfies the strong meromorphic approximation property in X. More generally every open set D of a reduced Stein space X of dimension 1 satisfies the strong meromorphic approximation property in X (see Corollary 5.3).

On the other hand a Stein open set D of a reduced Stein space X is meromorphically $\mathcal{O}(X)$-convex if and only if for every holomorphic function

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$\varphi \in \mathcal{O}(D)$, for every compact set K of D and for every $\varepsilon > 0$ there exist holomorphic functions $f, g \in \mathcal{O}(X)$ such that $g \not\equiv 0$ on any irreducible component of X, $g \neq 0$ on K and $\|\varphi - (f/g)\|_K < \varepsilon$ (see Lemma 2.1).

An open set D of $\mathbb{C}^n$ is meromorphically $\mathcal{O}(\mathbb{C}^n)$-convex if and only if D is rationally convex. By giving counterexamples we prove that a rationally convex open set D of $\mathbb{C}^n$, where $n \geq 2$, does not satisfy in general the strong meromorphic approximation property in $\mathbb{C}^n$ (see Propositions 4.1 and 4.2). We classify Stein open sets in $\mathbb{C}^n$ from the point of view of the approximation property (see Theorem 3.2).

2. Preliminaries

Throughout this paper all complex spaces are supposed to be *reduced* and *second countable*. Let X be a complex space. We denote by Ac the sheaf on X of germs of active holomorphic functions (see Grauert-Remmert [10, p. 97]). Then $\mathsf{Ac}(X)$ is the set of all $f \in \mathcal{O}(X)$ such that $f \not\equiv 0$ on any irreducible component of X. Let

$$\mathcal{Q}_X(D) := \{(f/g)\,|_D \mid f \in \mathcal{O}(X),\ g \in \mathsf{Ac}(X),\ g \neq 0 \text{ on } D\}$$

for every open set D of X. If X is a locally irreducible complex space in which every strong Poincaré problem is solvable (see Kaup-Kaup [15, p. 249]), then we have that $\mathcal{Q}_X(D) = \mathcal{M}(X) \cap \mathcal{O}(D)$ for every open set D of X.

Let X be a complex space and let $\mathcal{F} \subset \mathcal{O}(X)$. Then X is said to be *$\mathcal{F}$-convex* if for every compact set K of X the *holomorphically convex hull*

$$\hat{K}_{\mathcal{F}} := \{x \in X \mid |f(x)| \leq \|f\|_K \text{ for every } f \in \mathcal{F}\}$$

of K with respect to $\mathcal{F}$ is compact.

On the other hand a complex space X is said to be *meromorphically $\mathcal{F}$-convex* if for every compact set K of X the *meromorphically convex hull*

$$\tilde{K}_{\mathcal{F}} := \{x \in X \mid f(x) \in f(K) \text{ for every } f \in \mathcal{F}\}$$

of K with respect to $\mathcal{F}$ is compact. The set $\tilde{K}_X = {}_H K_X := \tilde{K}_{\mathcal{O}(X)}$ is said to be the *meromorphically convex hull* of K in X (cf. Hirschowitz [14, p. 49], Lupacciolu [16], Colţoiu [6], Abe-Furushima [4] and Abe [1, 2, 3]).

An open set D of a complex space X is said to be *meromorphically $\mathscr{F}$-convex* if D is meromorphically $\mathscr{F}|_D$-convex, that is, for every compact set K of D the set $\tilde{K}_{\mathscr{F}} \cap D$ is compact. We have the following characterizations of meromorphically $\mathscr{O}(X)$-convex open sets in a Stein space X.

Lemma 2.1 (Abe [1, Theorem 12]) *Let X be a Stein space and D an open set of X. Then the following four conditions are equivalent.*

(1) *D is meromorphically $\mathscr{O}(X)$-convex.*

(2) *For every compact set $K \subset D$ we have that $\tilde{K}_X \subset D$.*

(3) *For every compact set $K \subset D$ we have that $\tilde{K}_X = \tilde{K}_D$.*

(4) *For every compact set K of D the set $\tilde{K}_D$ is compact and for every holomorphic function $\varphi \in \mathscr{O}(D)$, for every compact set K of D and for every $\varepsilon > 0$ there exist holomorphic functions $f \in \mathscr{O}(X)$ and $g \in \mathrm{Ac}(X)$ such that $g \neq 0$ on K and $\|\varphi - (f/g)\|_K < \varepsilon$.*

Let X be a complex space. Let $f_1, f_2, \ldots, f_m \in \mathscr{O}(X)$ and $g_1, g_2, \ldots, g_m \in \mathrm{Ac}(X)$. Let $A := \{g_1 g_2 \cdots g_m = 0\}$. Let G be an open set of $X \setminus A$. Let $h_\mu := f_\mu / g_\mu$ for $\mu = 1, 2, \ldots, m$. Let $Z_1, Z_2, \ldots, Z_m$ be open sets of $\mathbb{C}$. Let

$$W := G \cap \{x \in X \setminus A \mid h_\mu(x) \in Z_\mu \text{ for every } \mu = 1, 2, \ldots, m\}$$

and assume that $W \Subset G$. Then the open set W is said to be a *meromorphic polyhedron* of X (see Abe [1, p. 266]). We use this notation for W in the following lemma.

Lemma 2.2. *Let X be a Stein space and W a meromorphic polyhedron of X with $Z_1 = Z_2 = \cdots = Z_m = \Delta$, where $\Delta := \{t \in \mathbb{C} \mid |t| < 1\}$. Then for every compact set K of W and for every $\varphi \in \mathscr{O}(W)$ there exist $u \in \mathscr{O}(X)$ and a monic monomial v of $g_1, g_2, \ldots, g_m$ such that $\|\varphi - (u/v)\|_K < \varepsilon$.*

Proof. There exist $n \in \mathbb{N}$ and $\theta_1, \theta_2, \ldots, \theta_n \in \mathscr{O}(X)$ such that the restriction $\psi_{W, \Delta^m \times \mathbb{C}^n} : W \to \Delta^m \times \mathbb{C}^n$ is a closed holomorphic embedding, where

$$\psi := (h_1, h_2, \ldots, h_m, \theta_1, \theta_2, \ldots, \theta_n) : X \setminus A \to \mathbb{C}^{m+n}$$

(see Abe [1, Lemma 8]). Since $\psi(W)$ is an analytic set of a Stein manifold $\Delta^m \times \mathbb{C}^n$ and the function $\varphi \circ \left(\psi_{W,\psi(W)}\right)^{-1} : \psi(W) \to \mathbb{C}$ is holomorphic, there exists $\alpha \in \mathcal{O}(\Delta^m \times \mathbb{C}^n)$ such that $\alpha = \varphi \circ \left(\psi_{W,\psi(W)}\right)^{-1}$ on $\psi(W)$. By considering the Taylor expansion of α at the origin there exists a polynomial function β on $\mathbb{C}^{m+n}$ such that $\|\alpha - \beta\|_{\psi(K)} < \varepsilon$. Since $\beta \circ \psi$ is a polynomial of $h_1, h_2, \ldots, h_m, \theta_1, \theta_2, \ldots, \theta_n$, there exist a polynomial u of $f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_n, \theta_1, \theta_2, \ldots, \theta_n$ and a monic monomial v of $g_1, g_2, \ldots, g_m$ such that $\beta \circ \psi = u/v$ on $X \setminus A$. Then we have that $\|\varphi - (u/v)\|_K < \varepsilon$. $\square$

For every open set D of a complex space X the topology of uniform convergence on compact sets gives the linear space $\mathcal{O}(D)$ the structure of Fréchet space (see Kaup-Kaup [15, E. 55j]). We say that an open set D of a complex space X satisfies the *strong meromorphic approximation property* in X if the set $\mathcal{Q}_X(D)$ is dense in $\mathcal{O}(D)$, that is, for every holomorphic function $\varphi \in \mathcal{O}(D)$, for every compact set K of D and for every $\varepsilon > 0$ there exist holomorphic functions $f \in \mathcal{O}(X)$ and $g \in \mathrm{Ac}(X)$ such that $g \neq 0$ on D and $\|\varphi - (f/g)\|_K < \varepsilon$.

Lemma 2.3. *Let X be a Stein space and D an open set of X. Then the following two conditions are equivalent.*

(1) *D is $\mathcal{Q}_X(D)$-convex.*

(2) *D is Stein and $\mathcal{Q}_X(D)$ is dense in $\mathcal{O}(D)$.*

Proof. (1) $\Rightarrow$ (2). Since $\mathcal{Q}_X(D) \subset \mathcal{O}(D)$, we have that $\hat{K}_D \subset \hat{K}_{\mathcal{Q}_X(D)}$ for every compact set K of D, where $\hat{K}_D := \hat{K}_{\mathcal{O}(D)}$. Since by assumption $\hat{K}_{\mathcal{Q}_X(D)}$ is compact, the set $\hat{K}_D$ is also compact. It follows that D is Stein. Take an arbitrary $\varphi \in \mathcal{O}(D)$. Let K be a compact set of D and let $\varepsilon > 0$. Since $\hat{K}_{\mathcal{Q}_X(D)}$ is compact, there exists an open set E of X such that $\hat{K}_{\mathcal{Q}_X(D)} \subset E \Subset D$. Take an arbitrary point $p \in \partial E$. Since $p \notin \hat{K}_{\mathcal{Q}_X(D)}$, there exist $f^{(p)} \in \mathcal{O}(X)$ and $g^{(p)} \in \mathrm{Ac}(X)$ such that $g^{(p)} \neq 0$ on D and $\left|h^{(p)}(p)\right| > \left\|h^{(p)}\right\|_K$, where $h^{(p)} := f^{(p)}/g^{(p)}$. Replacing $f^{(p)}$ by $f^{(p)}/c$, where $\left|h^{(p)}(p)\right| > c > \left\|h^{(p)}\right\|_K$, we have that $\left|h^{(p)}(p)\right| > 1 > \left\|h^{(p)}\right\|_K$. Then $V_p := \left\{x \in D \mid \left|h^{(p)}(x)\right| > 1\right\}$ is an open neighborhood of p. Since

∂E is compact, there exist finitely many points $p_1, p_2, \ldots, p_m \in \partial E$ such that $\partial E \subset \bigcup_{\mu=1}^{m} V_{p_\mu}$. Let $f_\mu := f^{(p_\mu)}$, $g_\mu := g^{(p_\mu)}$ and $h_\mu := f_\mu / g_\mu$ for $\mu = 1, 2, \ldots, m$. Let $A := \{g_1 g_2 \cdots g_m = 0\}$. Then the set

$$W := E \cap \{x \in X \setminus A \mid |h_\mu(x)| < 1 \text{ for every } \mu = 1, 2, \ldots, m\}$$

is a meromorphic polyhedron of X with $Z_1 = Z_2 = \cdots = Z_m = \Delta$ and we have that $K \subset W \Subset E$. By Lemma 2.2 there exist $u \in \mathcal{O}(X)$ and a monic monomial v of $g_1, g_2, \ldots, g_m$ such that $\|\varphi - (u/v)\|_K < \varepsilon$. Since $u/v \in \mathcal{Q}_X(D)$, the proof of the denseness of $\mathcal{Q}_X(D)$ in $\mathcal{O}(D)$ completes.
(2) $\Rightarrow$ (1). Since $\mathcal{Q}_X(D)$ is dense in $\mathcal{O}(D)$, we have that $\hat{K}_{\mathcal{Q}_X(D)} = \hat{K}_D$ for every compact set K of D. Since D is Stein, the set $\hat{K}_D$ is compact. It follows that D is $\mathcal{Q}_X(D)$-convex. $\square$

Proposition 2.4. *Let X be a complex space and D an open set of X. If D is $\mathcal{Q}_X(D)$-convex, then D is meromorphically $\mathcal{O}(X)$-convex.*

Proof. Take an arbitrary compact set K of D. Let $p \in D \setminus \hat{K}_{\mathcal{Q}_X(D)}$. There exist $u \in \mathcal{O}(X)$ and $v \in \mathrm{Ac}(X)$ such that $v \not\equiv 0$ on D and $|m(p)| > \|m\|_K$, where $m := u/v$. Let $h := m(p)v - u$. Then $h \in \mathcal{O}(X)$ and $h(p) = 0$. Assume that there exists a point $y \in K$ such that $h(y) = 0$. Then we have that $|m(p)| = |m(y)| \leq \|m\|_K$, which is a contradiction. It follows that $0 \notin h(K)$ and thus we have that $p \notin \tilde{K}_X$. Therefore $\tilde{K}_X \cap D \subset \hat{K}_{\mathcal{Q}_X(D)}$. Since $\hat{K}_{\mathcal{Q}_X(D)}$ is compact, the closed set $\tilde{K}_X \cap D$ of D is also compact. Thus we proved that D is meromorphically $\mathcal{O}(X)$-convex. $\square$

The converse of Proposition 2.4 is not true in general. We have the following example.

Example 2.1. Let $\mathbf{P}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let

$$X := (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbf{P}^1),$$

which is an analytic set of $\mathbb{C} \times \mathbf{P}^1$ and is neither Stein nor irreducible. Let

$$D := \{z \in \mathbb{C} \mid 0 < |z| < 1\} \times \{0\},$$

which is an analytic polyhedron of X. Let $K := \{z \in \mathbb{C} \mid |z| = 1/2\} \times \{0\}$. If $g \in \mathrm{Ac}(X)$ and $g \not\equiv 0$ on D, then $g \not\equiv 0$ on $\{z \in \mathbb{C} \mid |z| < 1\} \times \{0\}$.

We have that $\{z \in \mathbb{C} \mid 0 < |z| \leq 1/2\} \times \{0\} \subset \hat{K}_{\mathscr{Q}_X(D)}$ by the maximum modulus principle and $\hat{K}_{\mathscr{Q}_X(D)}$ is not compact. It follows that D is not $\mathscr{Q}_X(D)$-convex. However the open set D is meromorphically $\mathscr{O}(X)$-convex (see Abe [1, Proposition 4]).

Even if X is an irreducible Stein space, the converse of Proposition 2.4 is not true in general (see Theorem 3.2 in Sect. 3 and Propositions 4.1 and 4.2 in Sect. 4). On the other hand an open set D of a Stein space X is meromorphically $\mathscr{O}(X)$-convex if and only if D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of X such that D_ν is $\mathscr{Q}_X(D_\nu)$-convex for every $\nu \in \mathbb{N}$ (see Abe [2, Theorem 4.1]).

3. Classification of Stein open sets of $\mathbb{C}^n$

Let $z_1, z_2, \ldots, z_n$ be the standard coordinates of $\mathbb{C}^n$. As usual we denote by $\mathbb{C}[z_1, z_2, \ldots, z_n]$ and by $\mathbb{C}(z_1, z_2, \ldots, z_n)$ the set of polynomial functions on $\mathbb{C}^n$ and the set of rational functions on $\mathbb{C}^n$ respectively. We let

$$\mathscr{R}(D) := \mathbb{C}(z_1, z_2, \ldots, z_n) \cap \mathscr{O}(D)$$

for every open set D of $\mathbb{C}^n$.

For every compact set K of $\mathbb{C}^n$ the set $\tilde{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]}$ is said to be the *rationally convex hull* of K (cf. Stolzenberg [28, p. 262] and Gamelin [9, p. 69]).

An open set D of $\mathbb{C}^n$ is said to be *rationally convex* in $\mathbb{C}^n$ if D is meromorphically $\mathbb{C}[z_1, z_2, \ldots, z_n]$-convex. Since $\tilde{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]} = \tilde{K}_{\mathbb{C}^n}$ for every compact set K of $\mathbb{C}^n$, an open set D of $\mathbb{C}^n$ is rationally convex if and only if D is meromorphically $\mathscr{O}(\mathbb{C}^n)$-convex (see Abe [1, p. 265]).

We have the following lemma, the proof of which is not difficult and is omitted.

Lemma 3.1. *Let $n_1, n_2 \in \mathbb{N}$ and let $n := n_1 + n_2$. Let D_ν be an open set of $\mathbb{C}^{n_\nu}$ for each $\nu = 1, 2$ and let $D := D_1 \times D_2 \subset \mathbb{C}^n$. Then D is polynomially convex (resp. $\mathscr{R}(D)$-convex, $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex or rationally convex) if and only if D_ν is polynomially convex (resp. $\mathscr{R}(D_\nu)$-convex, $\mathscr{Q}_{\mathbb{C}^{n_\nu}}(D_\nu)$-convex or rationally convex) for each $\nu = 1, 2$.*

We have the following theorem classifying Stein open sets of $\mathbb{C}^n$ from the point of view of the approximation property.

Theorem 3.2. *Let D be an open set of $\mathbb{C}^n$. We have the following four inclusions.*

(a) *If D is polynomially convex, then D is $\mathscr{R}(D)$-convex.*

(b) *If D is $\mathscr{R}(D)$-convex, then D is $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex.*

(c) *If D is $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex, then D is rationally convex.*

(d) *If D is rationally convex, then D is Stein.*

If $n \geq 2$, then none of the converses of the four inclusions (a), (b), (c) *and* (d) *is true.*

Proof. Since $\hat{K}_{\mathscr{Q}_{\mathbb{C}^n}(D)} \subset \hat{K}_{\mathscr{R}(D)} \subset \hat{K}_{\mathbb{C}[z_1, z_2, \ldots, z_n]}$ for every compact set K of D, we have the inclusions (a) and (b). By Proposition 2.4 we have the inclusion (c). The inclusion (d) is well-known (see Abe [1, Corollary 13] in more general situation). Let $D_2 \subset \mathbb{C}^2$ be one of the Examples 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 and 4.7 in Sect. 4. Then by Lemma 3.1 the open set $D := D_2 \times \mathbb{C}^{n-2}$ of $\mathbb{C}^n$, $n \geq 2$, gives an example which shows that the converse of the inclusion (a), (b), (c) or (d) is not true. □

4. Examples

In this section we always denote by z and w the coordinates of $\mathbb{C}^2$.

Example 4.1. The *Hartogs triangle*

$$D := \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}$$

is an open set of $\mathbb{C}^2$ which is $\mathscr{R}(D)$-convex but not polynomially convex. This example D is not simply connected.

Example 4.2. The *Nishino domain*

$$D := \{(z, w) \in \mathbb{C}^2 \mid 1 < |z| < M,\ |w| < 1\} \setminus S,$$

where $S := \bigcup_{0 \leq t \leq 1} \{(z, w) \in \mathbb{C}^2 \mid (1-t)\, z^2 - 2tz + w = 0\}$ and $M > 1$, is $\mathscr{R}(D)$-convex. Nishino [19, 21] proved that if M is sufficiently large, then D is simply connected and is not polynomially convex.

Example 4.3. Let S be an irreducible transcendental hypersurface of $\mathbb{C}^2$ and let $D := \mathbb{C}^2 \setminus S$. Then D is $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex and is not $\mathscr{R}(D)$-convex (cf. Nishino [20, p. 99]). This example D is not simply connected.

Problem 4.1. Does there exist a simply connected open set D of $\mathbb{C}^n$ such that D is $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex but not $\mathscr{R}(D)$-convex?

Example 4.4. Let

$$D := (\mathbb{C}^* \times \mathbb{C}) \setminus S, \text{ where } S := \left\{(z, w) \in \mathbb{C}^* \times \mathbb{C} \mid w - \mathrm{e}^{1/z} = 0\right\}.$$

Then we have the following proposition.

Proposition 4.1. *The open set $D = (\mathbb{C}^* \times \mathbb{C}) \setminus S$ above is rationally convex in $\mathbb{C}^2$ and is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex.*

Proof. First we prove that D is rationally convex. Take an arbitrary compact set K of D. Let K_1 be the image of K by the projection $\mathbb{C}^2 \to \mathbb{C}$, $(z, w) \mapsto z$. Then $\varepsilon := \min_{(z,w)\in K} |w - \mathrm{e}^{1/z}| > 0$ and $\delta := \min_{z\in K_1} |z| > 0$. Let $F_n(z) := \sum_{k=0}^{n-1} z^{n-k}/k! \in \mathbb{C}[z]$ and $f_n(z, w) := z^n w - F_n(z) \in \mathbb{C}[z, w]$ for every $n \in \mathbb{N}$. Since the sequence of functions $F_n(z)/z^n = \sum_{k=0}^{n-1} (1/z)^k /k!$, $n \in \mathbb{N}$, tends to the function $\mathrm{e}^{1/z}$ on any compact set of $\mathbb{C}^*$, there exists $N_1 \in \mathbb{N}$ such that $|F_n(z)/z^n - \mathrm{e}^{1/z}| < \varepsilon/2$ for every $z \in K_1$ and $n \geq N_1$. Take an arbitrary $\xi \in \mathbb{C}^*$. We have that

$$\begin{aligned}\left|\mathrm{e}^{1/\xi} - \frac{F_n(\xi)}{\xi^n}\right| &= \left|\sum_{k=0}^{\infty} \frac{1}{(n+k)!}\left(\frac{1}{\xi}\right)^{n+k}\right| \leq \sum_{k=0}^{\infty} \frac{1}{(n+k)!}\left(\frac{1}{|\xi|}\right)^{n+k} \\ &< \sum_{k=0}^{\infty} \frac{1}{n!\,k!}\left(\frac{1}{|\xi|}\right)^{n+k} = \frac{\mathrm{e}^{1/|\xi|}}{n!\,|\xi|^n}.\end{aligned}$$

It follows that $|f_n(\xi, \mathrm{e}^{1/\xi})| = |\xi^n \mathrm{e}^{1/\xi} - F_n(\xi)| < \mathrm{e}^{1/|\xi|}/n!$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} (1/\delta)^n /n! = 0$, there exists $N_2 \in \mathbb{N}$ such that $(1/\delta)^n /n! <$

$\varepsilon\,\mathrm{e}^{-1/|\xi|}/2$ for every $n \geq N_2$. If $(z,w) \in K$ and $n \geq N := \max\{N_1, N_2\}$, then

$$|f_n(z,w)| = \left|z^n\left(w - \frac{F_n(z)}{z^n}\right)\right| \geq |z^n|\left(\left|w - \mathrm{e}^{1/z}\right| - \left|\frac{F_n(z)}{z^n} - \mathrm{e}^{1/z}\right|\right)$$
$$> \delta^n\left(\varepsilon - \frac{\varepsilon}{2}\right) = \frac{\varepsilon\delta^n}{2} > \frac{\mathrm{e}^{1/|\xi|}}{n!} > \left|f_n(\xi, \mathrm{e}^{1/\xi})\right|.$$

Therefore $\left|f_N(\xi, \mathrm{e}^{1/\xi})\right| < \min_{(z,w)\in K} |f_N(z,w)|$ and we have that $(\xi, \mathrm{e}^{1/\xi}) \notin \tilde{K}_{\mathbb{C}^2}$. Thus we proved that $\tilde{K}_{\mathbb{C}^2} \cap S = \emptyset$. On the other hand it is clear that $\tilde{K}_{\mathbb{C}^2} \cap (\{0\} \times \mathbb{C}) = \emptyset$. It follows that $\tilde{K}_{\mathbb{C}^2} \subset D$ and the proof of the rational convexity of D completes (see Lemma 2.1). Next we prove that D is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex. Since the function $z \mapsto \mathrm{e}^{1/z}$ has an essential singularity at the origin $z = 0$, we have that $\overline{S} = S \cup (\{0\} \times \mathbb{C})$, where $\overline{S}$ is the closure of S in $\mathbb{C}^2$. Assume that $\overline{S}$ is an analytic set of $\mathbb{C}^2$. Then $\overline{S}$ is an irreducible curve in $\mathbb{C}^2$ since S is connected and non-singular. On the other hand we have that $\{0\} \times \mathbb{C} \subsetneq S$. It contradicts the identity theorem for analytic sets (see Grauert-Remmert [10, p. 167]). It follows that $\overline{S}$ is not an analytic set of $\mathbb{C}^2$. Take a point $\xi \in \mathbb{C}^*$. Then $L := \{\xi\} \times \left\{w \in \mathbb{C} \mid \left|w - \mathrm{e}^{1/\xi}\right| = 1\right\}$ is a compact set of D. Take an arbitrary $h = f/g \in \mathscr{Q}_{\mathbb{C}^2}(D)$, where $f, g \in \mathscr{O}(\mathbb{C}^2)$ and $g \neq 0$ on D. Assume that $\{g = 0\} \cap (\mathbb{C}^* \times \mathbb{C}) \neq \emptyset$. Since S is irreducible and $\{g = 0\} \cap (\mathbb{C}^* \times \mathbb{C}) \subset S$, we have that $\{g = 0\} \cap (\mathbb{C}^* \times \mathbb{C}) = S$ by the identity theorem for analytic sets. Then we obtain that $\overline{S} = \{g = 0\}$. Since $\overline{S}$ is not an analytic set of $\mathbb{C}^2$, it is a contradiction. It follows that $g \neq 0$ on $\mathbb{C}^* \times \mathbb{C}$ and the function h is holomorphic on $\mathbb{C}^* \times \mathbb{C}$. By the maximum modulus principle we have that $|h(\xi, w)| \leq \|h\|_L$ if $\left|w - \mathrm{e}^{1/\xi}\right| \leq 1$. Therefore $\{\xi\} \times \left\{w \in \mathbb{C} \mid 0 < \left|w - \mathrm{e}^{1/\xi}\right| \leq 1\right\} \subset \hat{L}_{\mathscr{Q}_{\mathbb{C}^2}(D)}$. Since $(\xi, \mathrm{e}^{1/\xi}) \notin D$, the set $\hat{L}_{\mathscr{Q}_{\mathbb{C}^2}(D)}$ is not compact. Thus we proved that D is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex. □

Example 4.5. Let $D_1 := \Delta \times \mathbb{C}^*$ and $D_2 := \left(\mathbb{C} \setminus \overline{\Delta}\right) \times \mathbb{C}$, where Δ denotes the unit disk in $\mathbb{C}$. Although the open set D_ν is $\mathscr{Q}_{\mathbb{C}^2}(D_\nu)$-convex for each $\nu = 1, 2$, the disjoint union $D := D_1 \cup D_2$ is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex by the following proposition.

Proposition 4.2. *The open set $D = D_1 \cup D_2$ above is rationally convex in $\mathbb{C}^2$ and is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex.*

Proof. First we prove that D is rationally convex. Take an arbitrary compact set K of D. There exist numbers $a \in (0,1)$, $b > 0$, $c > 1$ and $d > 0$ such that $K_1 := K \cap D_1 \subset \{(z,w) \in \mathbb{C}^2 \mid |z| \leq a, |w| \geq b\}$ and $K_2 := K \cap D_2 \subset \{(z,w) \in \mathbb{C}^2 \mid |z| \geq c, |w| \leq d\}$. Take $N_1 \in \mathbb{N}$ such that $a^{N_1} < b/2$ and $c^{N_1} > b/2 + d$. Let $f_n(z,w) := w - z^n \in \mathbb{C}[z,w]$ for every $n \in \mathbb{N}$. If $n \geq N_1$, then $|f_n(z,w)| \geq |w| - |z|^n \geq b - a^n \geq b - b/2 = b/2$ for every $(z,w) \in K_1$ and $|f_n(z,w)| \geq |z|^n - |w| \geq c^n - d \geq (b/2+d) - d = b/2$ for every $(z,w) \in K_2$. It follows that $\min_{(z,w)\in K} |f_n(z,w)| > b/2$ if $n \geq N_1$. Let $\xi \in \Delta$ and take $N \geq N_1$ such that $|f_N(\xi,0)| = |\xi|^N < b/2$. Then $|f_N(\xi,0)| < \min_{(z,w)\in K} |f_N(z,w)|$. Therefore $(\xi,0) \notin \tilde{K}_{\mathbb{C}^2}$ for every $\xi \in \Delta$ and thus $\tilde{K}_{\mathbb{C}^2} \cap (\Delta \times \{0\}) = \emptyset$. Let $(\xi,\eta) \in \partial\Delta \times \mathbb{C}$ and $f(z,w) := z - \xi \in \mathbb{C}[z,w]$. Since $f(\xi,\eta) = 0 \notin f(K)$, we have that $(\xi,\eta) \notin \tilde{K}_{\mathbb{C}^2}$. Therefore $\tilde{K}_{\mathbb{C}^2} \cap (\partial\Delta \times \mathbb{C}) = \emptyset$. Since $\mathbb{C}^2 \setminus D = (\Delta \times \{0\}) \cup (\partial\Delta \times \mathbb{C})$, we have that $\tilde{K}_{\mathbb{C}^2} \subset D$. It follows that D is rationally convex (see Lemma 2.1). Next we prove that D is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex. The set $L := \{0\} \times \{w \in \mathbb{C} \mid |w| = 1\}$ is compact and is contained in D. Take an arbitrary $h = f/g \in \mathscr{Q}_{\mathbb{C}^2}(D)$, where $f, g \in \mathscr{O}(\mathbb{C}^2)$ and $g \neq 0$ on D. Since $g \neq 0$ on $(\mathbb{C} \setminus \overline{\Delta}) \times \{0\}$, the number of the zero points of the function $z \mapsto g(z,0)$ is finite and there exists $r \in (0,1)$ such that $g(z,0) \neq 0$ for $r \leq |z| < 1$. On the other hand the function $z \mapsto g(z,w)$ has no zero points in Δ if $w \neq 0$. Therefore by the Hurwitz theorem the function $z \mapsto g(z,0)$ has no zero points in the disk $\{z \in \mathbb{C} \mid |z| < r\}$. It follows that $g \neq 0$ on $\Delta \times \mathbb{C}$ and h is holomorphic on $\Delta \times \mathbb{C}$. By the maximum modulus principle we have that $|h(0,w)| \leq \|h\|_L$ if $|w| \leq 1$. Therefore $\{0\} \times \{w \in \mathbb{C} \mid 0 < |w| \leq 1\} \subset \hat{L}_{\mathscr{Q}_{\mathbb{C}^2}(D)}$. Since $(0,0) \notin D$, the set $\hat{L}_{\mathscr{Q}_{\mathbb{C}^2}(D)}$ is not compact. Thus we proved that D is not $\mathscr{Q}_{\mathbb{C}^2}(D)$-convex. □

Problem 4.2. Does there exist a simply connected open set D of $\mathbb{C}^n$ such that D is rationally convex but not $\mathscr{Q}_{\mathbb{C}^n}(D)$-convex?

Example 4.6. Stein [26, p. 757] gave the following example. Let

$$D := (\mathbb{C}^*)^2 \setminus A, \text{ where } A := \left\{(z,w) \in (\mathbb{C}^*)^2 \mid z = w^{\mathrm{i}}\right\}.$$

Then D is a Stein open set of $\mathbb{C}^2$. Let r and R be numbers such that

$e^{-\pi} < r < 1 < R < e^{\pi}$. Let

$$\begin{aligned}
&\alpha_1 := \left\{ e^{i\theta} \mid 0 \le \theta \le \pi \right\}, \ \alpha_2 := \left\{ e^{i\theta} \mid -\pi \le \theta \le 0 \right\}, \\
&\beta_1 := \{ w \in \mathbb{C} \mid |w| = r \}, \ \beta_2 := \{ w \in \mathbb{C} \mid |w| = R \}, \\
&\Gamma := \{ w \in \mathbb{C} \mid r \le |w| \le R \}, \text{ and} \\
&K := (\alpha_1 \times \beta_1) \cup (\alpha_2 \times \beta_2) \cup (\{-1\} \times \Gamma).
\end{aligned}$$

Then K is a compact set of D and we have that $\{1\} \times \Gamma \subset \tilde{K}_{\mathbb{C}^2}$ (see Proposition 4.3 below). Since $(1,1) \in A \cap (\{1\} \times \Gamma)$, we have that $\tilde{K}_{\mathbb{C}^2} \not\subset D$. It follows that D is not rationally convex in $\mathbb{C}^2$ (see Lemma 2.1). This example D is not simply connected. Oka [23] also gave a similar example (see Nishino [20, p. 99]).

Proposition 4.3. *For the sets Γ and K above we have that $\{1\} \times \Gamma \subset \tilde{K}_{\mathbb{C}^2}$.*

Proof. Take an arbitrary $f \in \mathcal{O}(\mathbb{C}^2)$ such that $f \neq 0$ on K. Since the function

$$N_\nu(z) := \frac{1}{2\pi i} \int_{\beta_\nu} \frac{f_w(z,w)}{f(z,w)} \, dw$$

is continuous and with discrete values on the connected set α_ν, it must be constant on α_ν for each $\nu = 1, 2$. On the other hand we have that

$$N_2(-1) - N_1(-1) = \frac{1}{2\pi i} \int_{\beta_2 - \beta_1} \frac{f_w(-1,w)}{f(-1,w)} \, dw = 0$$

because $f(-1,w) \neq 0$ for every $w \in \Gamma$. It follows that

$$N_1(1) = N_1(-1) = N_2(-1) = N_2(1)$$

and thus we have that

$$\frac{1}{2\pi i} \int_{\beta_2 - \beta_1} \frac{f_w(1,w)}{f(1,w)} \, dw = 0.$$

Therefore by the argument principle the function $f(1,w)$ of w has no zero points in Γ. Thus we proved that $f \neq 0$ on $\{1\} \times \Gamma$ for every $f \in \mathcal{O}(\mathbb{C}^2)$ such that $f \neq 0$ on K. This means that $\{1\} \times \Gamma \subset \tilde{K}_{\mathbb{C}^2}$. $\square$

Example 4.7. Let Δ denote the unit disk. Wermer [29] gave an example of an open set of $\mathbb{C}^3$ which is biholomorphic to Δ^3 and is not rationally convex in $\mathbb{C}^3$ (see Stolzenberg [27]). Wermer [30] also gave a similar example biholomorphic to Δ^2, which is as follows. Let

$$K := \{(z, w) \in \mathbb{C}^2 \mid w = \overline{z},\ |\mathrm{Re}(z)| \le 1,\ |\mathrm{Im}(z)| \le 1\} \text{ and}$$
$$\psi : \mathbb{C}^2 \to \mathbb{C}^2,\ \psi(z, w) := (z, (1 + \mathrm{i})\, w - \mathrm{i}\, zw^2 - z^2 w^3).$$

By Wermer [30] there exists an open neighborhood U of K such that U is biholomorphic to Δ^2, the set $D := \psi(U)$ is open in $\mathbb{C}^2$, the restriction map $\psi_{U,D} : U \to D$ is biholomorphic and $(1/2, 0) \notin D$ whereas $\partial\Delta \times \{0\} \subset \psi(K)$ (see also Fornæss-Stensønes [8, pp. 212–213] and Ohsawa [22, p. 81]). Then the open set D is not rationally convex in $\mathbb{C}^2$ because D is simply connected and $D \cap (\mathbb{C} \times \{0\})$ is not simply connected (see Nishino [20, Remark 3.6] and Abe [3, Corollary 6]). Especially D is not polynomially convex, which is the original assertion of Wermer [30].

5. Stein space of dimension 1

A complex space X of dimension 1 is Stein if and only if X has no compact irreducible component of dimension 1 by Narasimhan [18].

Lemma 5.1. *Let X be a Stein space of dimension 1. Then for every $p \in X$ there exists $g \in \mathcal{O}(X)$ such that $\{g = 0\} = \{p\}$.*

Proof. If p is an isolated point of X, then the assertion is clear. We consider the case when p is not an isolated point of X. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of irreducible components of X. Take a point $q_\lambda \in X_\lambda \setminus \{p\}$ for every $\lambda \in \Lambda$. Then the set $\{q_\lambda \mid \lambda \in \Lambda\} \cup \{p\}$ is discrete in X. Since X is Stein, there exists $\tau \in \mathcal{O}(X)$ such that $\tau \equiv 1$ on $\{q_\lambda \mid \lambda \in \Lambda\}$ and $\tau(p) = 0$. Since $\tau \not\equiv 0$ on X_λ for every $\lambda \in \Lambda$, we have that $\dim_p N(\tau) = 0$ by the active lemma (see Grauert-Remmert [10, p. 100]). It follows that there exists a neighborhood U of p such that $U \cap \{\tau = 0\} = \{p\}$. We have that $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) = 0$ because X has the homotopy type of a CW-complex of dimension ≤ 1 (see Hamm [11, 12] and Hamm-Mihalache [13]). It follows that there exist $g_0 \in \mathcal{O}^*(U)$ and $g_1 \in \mathcal{O}^*(X \setminus \{p\})$ such that

$\tau = g_1/g_0$ on $U \setminus \{p\}$. We define $g \in \mathscr{O}(X)$ by the equalities $g = \tau g_0$ on U and $g = g_1$ on $X \setminus \{p\}$. Then we have that $\{g = 0\} = \{p\}$. □

Theorem 5.2. *Let X be a Stein space of dimension* 1*. Then every open set D of X is $\mathscr{Q}_X(D)$-convex.*

Proof. Let K be a compact set of X. Take an arbitrary sequence $\{p_\nu\}_{\nu=1}^\infty$ of points of $\hat{K}_{\mathscr{Q}_X(D)}$. Since $\hat{K}_{\mathscr{Q}_X(D)} \subset \hat{K}_X \cap D$ and $\hat{K}_X$ is compact, we may assume without loss of generality that $\{p_\nu\}_{\nu=1}^\infty$ converges to a point $p \in \overline{D}$. Assume that $p \in \partial D$. Then p is not an isolated point of X. By Lemma 5.1 there exists $g \in \mathscr{O}(X)$ such that $\{g = 0\} = \{p\}$. Let $\delta := \min_{x \in K} |g(x)|$ and let $h := \delta/g$. We then have $\delta > 0$, $h \in \mathscr{Q}_X(D)$ and $\|h\|_K \leq 1$. If $x \in U := \{|g| < \delta\}$, then $|h(x)| > 1 \geq \|h\|_K$ and thus $x \notin \hat{K}_{\mathscr{Q}_X(D)}$. Therefore $\hat{K}_{\mathscr{Q}_X(D)} \cap U = \emptyset$. It contradicts the fact that p is an adherent point of $\hat{K}_{\mathscr{Q}_X(D)}$ in X. It follows that $p \in D$. Since $\hat{K}_{\mathscr{Q}_X(D)}$ is a closed set of D, we obtain that $p \in \hat{K}_{\mathscr{Q}_X(D)}$. Thus we proved that $\hat{K}_{\mathscr{Q}_X(D)}$ is compact. □

By the rational approximation theorem of Runge [25] (see Rudin [24, Theorem 13.9]) every holomorphic function f on an open set D of $\mathbb{C}$ can be uniformly approximated on every compact set K of D by rational functions which are holomorphic on D. If moreover D is simply connected, then every holomorphic function f on D can be uniformly approximated on every compact set K of D by polynomial functions.

As usual a non-compact connected complex manifold of dimension 1 is said to be an *open Riemann surface.* By Behnke-Stein [5, Satz 6] an open set D of an open Riemann surface X is $\mathscr{O}(X)$-convex if and only if no connected component of $X \setminus D$ is compact. Mihalache [17] generalized this result to Stein spaces of pure dimension 1. Colţoiu-Silva [7] obtained a generalization to complex spaces of pure dimension n with no compact irreducible components.

Behnke-Stein [5, Satz 13] also proved that every holomorphic function on an open set D of an open Riemann surface X can be uniformly approximated on every compact set K of D by meromorphic functions on X which are holomorphic on D and have at most finitely many poles on

∂D. As a corollary to Theorem 5.2 we have the following meromorphic approximation theorem in a Stein space of dimension 1.

Corollary 5.3. *Let X be a Stein space of dimension 1. Then every open set D of X satisfies the strong meromorphic approximation property in X, that is, for every $\varphi \in \mathscr{O}(D)$, for every compact set K of D and for every $\varepsilon > 0$ there exist $m \in \mathscr{M}(X) \cap \mathscr{O}(D)$ such that $\|\varphi - m\|_K < \varepsilon$.*

Proof. The assertion is a direct consequence of both Theorem 5.2 and Lemma 2.3. □

We also have the following weak version of the meromorphic approximation theorem (cf. Rudin [24, Theorem 13.6]).

Corollary 5.4. *Let X be a Stein space of dimension 1 and K a compact set of X. Then for every $\varphi \in \mathscr{O}(K)$ and for every $\varepsilon > 0$ there exist $m \in \mathscr{M}(X) \cap \mathscr{O}(K)$ such that $\|\varphi - m\|_K < \varepsilon$.*

Proof. Take an open set D of X such that $K \subset D$ and $\varphi \in \mathscr{O}(D)$. Then we have the assertion by Corollary 5.3 or by Lemma 2.1. □

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