A COHOMOLOGICAL CRITERION FOR SPLITTING OF VECTOR BUNDLES ON MULTIPROJECTIVE SPACE

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Abstract. This paper is devoted to the study of a cohomological criterion for the splitting of a vector bundle on multiprojective space. The criterion extends a result of Ballico-Malaspina towards a generalization of the Horrocks criterion on multiprojective space.

1. Introduction

This paper concerns a cohomological criterion for the splitting of vector bundles on multiprojective space. The Horrocks theorem [3] says that a vector bundle E on $\mathbb{P}_k^n$ is a direct sum of line bundles if and only if $\mathrm{H}^i(E(t)) = 0$ for all $1 \le i \le n-1$ and t. We will study vector bundles on $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$. In their paper [1, (1.3),(1.4)] Ballico and Malaspina give a cohomological criterion for vector bundles on multiprojective space, see also [2]. We will extend their result towards a framework of a generalization of Horrocks criterion.

Throughout this paper k is an algebraically closed field. Our theorem works on the multiprojective space $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$. For a coherent sheaf F on X, we write $F(a,b) = F \otimes p_1^*\mathcal{O}_{\mathbb{P}_k^{n_1}}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}_k^{n_2}}(b)$, where p_1 and p_2 are the first and second projections from $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$ to $\mathbb{P}_k^{n_1}$ and $\mathbb{P}_k^{n_2}$ respectively. Our main theorem is the following.

Theorem 1.1. *Let E be a vector bundle on $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$, where $n_1 \ge 2$ and $n_2 \ge 2$. The vector bundle E is a direct sum of line bundles of $\mathcal{O}_X$, $\mathcal{O}_X(0,1)$, $\mathcal{O}_X(0,2)$, $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(2,0)$ twisted by line bundles of the form $\mathcal{O}_X(\ell,\ell)$ if and only if*

$$\mathrm{H}^i(E(j_1+t, j_2+t)) = 0$$

for all integers i, j_1, j_2 and t satisfying that $1 \le i \le n_1+n_2-1$, $-i \le j_1+j_2 \le 0$, $-n_1 \le j_1 \le 0$ and $-n_2 \le j_2 \le 0$ except for $(i,j_1,j_2) = (n_1,-n_1,0),(n_1,-n_1+1,0),(n_2,0,-n_2),(n_2,0,-n_2+1)$.

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The theorem is obtained from an application of the multigraded Castelnuovo-Mumford regularity. This idea extends to a possible step towards a classification of ACM vector bundles on multiprojective space.

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2. Proof of the Main Theorem

We describe the multigraded Castelnuovo-Mumford regularity according to [1] with some remarks, see also [4]. This ingenious definition yields the corresponding property, which will play an important role for the proof of the main result.

Definition 2.1. Let $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$, where $n_1 \geq 0$ and $n_2 \geq 0$. A coherent sheaf F on X is said to be 0-regular if $\mathrm{H}^i(F(j_1, j_2)) = 0$ for all integers i, j_1 and j_2 such that $i \geq 1$, $j_1 + j_2 = -i$, $-n_1 \leq j_1 \leq 0$ and $-n_2 \leq j_2 \leq 0$.

Further, a coherent sheaf F on X is said to be (m_1, m_2)-regular if $F(m_1, m_2)$ is 0-regular.

Remark 2.2. Let F be a coherent sheaf on $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$. Assume that F is 0-regular. For a generic hyperplane H_1 of $\mathbb{P}_k^{n_1}$, $F|_{L_1}$ is 0-regular on $L_1 = H_1 \times \mathbb{P}_k^{n_2}$.

Remark 2.3. Let E be a vector bundle on $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$. Assume that E is 0-regular. Then $E(m_1, m_2)$ is 0-regular for $m_1 \geq 0$, $m_2 \geq 0$. Further, E is globally generated.

Now we will give the proof of our main theorem.

Proof of Theorem 1.1. In order to prove the "only if" part we need to consider the vanishing of the cohomologies $\mathrm{H}^i(\mathcal{O}_X(a,b) \otimes \mathcal{O}_X(j_1+t, j_2+t)) = 0$ for $(a,b) = (0,0), (1,0), (0,1), (2,0), (0,2)$. In fact, $\mathrm{H}^{n_1}(\mathcal{O}_X(j_1+t+a, j_2+t+b)) \neq 0$ if and only if $j_1+t+a \leq -n_1-1$ and $j_2+t+b \geq 0$, in other words, $-j_2-b \leq t \leq -n_1-j_1-a-1$. Similarly, $\mathrm{H}^{n_2}(\mathcal{O}_X(j_1+t+a, j_2+t+b)) \neq 0$ if and only if $-j_1 - a \leq t \leq -n_2 - j_2 - b - 1$. Thus we obtain that if $j_1 - n_2 + a - b \leq j_2 \leq j_1 + n_1 + a - b$ for $(a,b) = (0,0), (1,0), (0,1), (2,0), (0,2)$, all the required cohomologies vanish.

We will show the "if" part. There is an integer t such that $E(t,t)$ is 0-regular but $E(t-1, t-1)$ is not 0-regular.

First assume that $\mathrm{H}^{n_1+n_2}(E(-n_1+t-1, -n_2+t-1)) \neq 0$. Then we have $\mathrm{H}^0(E^\vee(-t,-t)) \neq 0$ by Serre duality, which gives a non-zero map $E(t,t) \to \mathcal{O}_X$. The (t,t)-regularity of E implies that $E(t,t)$ is globally generated. Then we have a non-zero map $\oplus \mathcal{O}_X \to E(t,t) \to \mathcal{O}_X$, which must be surjective and split. Hence $\mathcal{O}_X$ is a direct summand of $E(t,t)$.

Thus we may assume that $\mathrm{H}^{n_1+n_2}(E(-n_1+t-1, -n_2+t-1)) = 0$. Then we have only to focus on the case of non-vanishing of the n_1-th cohomologies of the vector bundle E with some twist. The rest of the cases, that is, that of the n_2-th cohomologies, are similarly proved.

Keeping in mind that E is (t,t)-regular but not $(t-1,t-1)$-regular, we have only to consider the case either $\mathrm{H}^{n_1}(E(-n_1+t,t-1)) \neq 0$ or $\mathrm{H}^{n_1}(E(-n_1+t-1,t-1)) \neq 0$ from the assumption. Then we divide into 3 cases:

i) $\mathrm{H}^{n_1}(E(-n_1+t,t-1)) \neq 0$,
ii) $\mathrm{H}^{n_1}(E(-n_1+t-1,t-2)) \neq 0$,
iii) $\mathrm{H}^{n_1}(E(-n_1+t,t-1)) = \mathrm{H}^{n_1}(E(-n_1+t-1,t-2)) = 0$ but $\mathrm{H}^{n_1}(E(-n_1+t-1,t-1)) \neq 0$.

For the case i), there is a nonzero element s of $\mathrm{H}^{n_1}(E(-n_1+t,t-1))$. Let R_u be the polynomial ring in n_u+1 variables over k for $u=1,2$. Let us take the Koszul complex

$$\mathbf{K_{u\bullet}} : 0 \to F_{u,n_u+1} \to F_{u,n_u} \to \cdots \to F_{u,r} \to \cdots \to F_{u,1} \to F_{u,0} \to 0,$$

where $F_{u,r}$ is a direct sum of ${}_{n_u+1}C_r$ copies of $R_u(-r)$ for $u=1,2$. Let us consider the exact sequence $p_1^*(\widetilde{\mathbf{K_{1\bullet}}}) \otimes E(t+1,t-1)$, that is, $0 \to E(-n_1+t,t-1) \to E(-n_1+t+1,t-1)^{\oplus n_1+1} \to \cdots \to E(t-r+1,t-1)^{\oplus_{n_1+1}C_r} \to \cdots \to E(t,t-1)^{\oplus n_1+1} \to E(t+1,t-1) \to 0$. In order to construct a surjective map

$$\varphi : \mathrm{H}^0(E(t+1,t-1)) \to \mathrm{H}^{n_1}(E(-n_1+t,t-1))$$

we need to show $\mathrm{H}^i(E(t-i+1,t-1)) = 0$ for $i=1,\cdots,n_1$. In fact, we see $\mathrm{H}^i(E(t-i+1,t-1)) = \mathrm{H}^i(E(-i+1,-1) \otimes \mathcal{O}_X(t,t)) = 0$ because $-i \leq (-i+1)+(-1) \leq 0$, $-n_1 \leq -i+1 \leq 0$ and $-n_2 \leq -1 \leq 0$. Thus there is a nonzero element $g \in \mathrm{H}^0(E(t+1,t-1))$ such that $\varphi(g) = s (\neq 0) \in \mathrm{H}^{n_1}(E(-n_1+t,t-1))$.

Let us consider the exact sequence $p_2^*(\widetilde{\mathbf{K_{2\bullet}}}) \otimes E^\vee(-t-1,-t+1)$, that is, $0 \to E^\vee(-t-1,-n_2-t) \to E^\vee(-t-1,-n_2-t+1)^{\oplus n_2+1} \to \cdots \to E^\vee(-t-1,-t-r+1)^{\oplus_{n_2+1}C_r} \to \cdots \to E^\vee(-t-1,-t)^{\oplus n_2+1} \to E^\vee(-t-1,-t+1) \to 0$. In order to construct a surjective map

$$\psi : \mathrm{H}^0(E^\vee(-t-1,-t+1)) \to \mathrm{H}^{n_2}(E^\vee(-t-1,-n_2-t))$$

we need to show $\mathrm{H}^i(E^\vee(-t-1,-t-i+1)) = 0$ for $i=1,\cdots,n_2$, equivalently $\mathrm{H}^i(E(-n_1+t,n_1+t-i-2)) = 0$ for $i=n_1,\cdots,n_1+n_2-1$ by Serre duality. In fact, we see $\mathrm{H}^i(E(-n_1+t,n_1+t-2-i)) = \mathrm{H}^i(E(-n_1+1,n_1-i-1) \otimes \mathcal{O}_X(t-1,t-1)) = 0$ because $-i \leq (-n_1+1)+(n_1-i-1) \leq 0$, $-n_1 \leq -n_1+1 \leq 0$ and $-n_2 \leq n_1-i-1 \leq 0$. By taking a dual element $s^* \in \mathrm{H}^{n_2}(E^\vee(-t-1,-n_2-t))$ corresponding to $s \in \mathrm{H}^{n_1}(E(-n_1+1,t-1))$, we have a nonzero element $f \in \mathrm{H}^0(E^\vee(-t-1,-t+1))$ such that $\psi(f) = s^*(\neq 0) \in \mathrm{H}^{n_2}(E^\vee(-t-1,-n_2-t))$. The elements g and f can be regarded as elements of $\mathrm{Hom}(\mathcal{O}_X(0,2), E(t+1,t+1))$ and $\mathrm{Hom}(E(t+1,t+1), \mathcal{O}_X(0,2))$ respectively. Let us consider the commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^0(E(t+1,t-1)) \otimes \mathrm{H}^0(E^\vee(-t-1,-t+1)) & \to & \mathrm{H}^0(\mathcal{O}_X) \\
\downarrow & & \downarrow \\
\mathrm{H}^{n_1}(E(t-n_1,t-1)) \otimes \mathrm{H}^{n_2}(E^\vee(-t-1,-t-n_2)) & \to & \mathrm{H}^{n_1+n_2}(\mathcal{O}_X(-n_1-1,-n_2-1)),
\end{array}$$

where the left vertical map is $\varphi \otimes \psi$, the right vertical isomorphism gives the canonical element. Thus we have that $f \circ g$ is an isomorphism. Hence $\mathcal{O}_X(0, 2)$ is a direct summand of $E(t + 1, t + 1)$.

For the case ii), there is a nonzero element s of $\mathrm{H}^{n_1}(E(-n_1 + t - 1, t - 2))$. Then we take the corresponding element s^* of $\mathrm{H}^{n_2}(E^\vee(-t, -n_2 - t + 1))$ by Serre duality. As in the case i) we will have surjective maps $\varphi : \mathrm{H}^0(E(t, t - 2)) \to \mathrm{H}^{n_1}(E(-n_1 + t - 1, t - 2))$ and $\psi : \mathrm{H}^0(E^\vee(-t, -t + 2)) \to \mathrm{H}^{n_2}(E^\vee(-t, -n_2 - t + 1))$. As in the same procedure by Koszul complex, we have only to prove that $\mathrm{H}^i(E(t - i, t - 2)) = 0$ for $1 \leq i \leq n_1$ and $\mathrm{H}^i(E(-n_1 + t - 1, n_1 + t - i - 3)) = 0$ for $n_1 \leq i \leq n_1 + n_2 - 1$. In fact, we see $\mathrm{H}^i(E(t - i, t - 2)) = \mathrm{H}^i(E(-i + 1, -1) \otimes \mathcal{O}_X(t - 1, t - 1)) = 0$ by assumption because $-i \leq (-i + 1) + (-1) \leq 0$, $-n_1 \leq -i + 1 \leq 0$ and $-n_2 \leq -1 \leq 0$. On the other hand, we see $\mathrm{H}^i(E(-n_1 + t - 1, n_1 + t - i - 3)) = \mathrm{H}^i(E(-n_1 + 1, n_1 - i - 1) \otimes \mathcal{O}_X(t - 2, t - 2)) = 0$ by assumption because $-i \leq (-n_1 + 1) + (n_1 - i - 1) \leq 0$, $-n_1 \leq -n_1 + 1 \leq 0$ and $-n_2 \leq n_1 - i - 1 \leq 0$. Then the maps g and f regarded as elements of $\mathrm{H}^0(E(t, t - 2))$ and $\mathrm{H}^0(E^\vee(-t, -t + 2))$ satisfying $\varphi(g) = s$ and $\psi(f) = s^*$ give a splitting map from $\mathcal{O}_X(-t, -t + 2)$ to E. Hence $\mathcal{O}_X(0, 2)$ is a direct summand of $E(t, t)$.

For the case iii), there is a nonzero element s of $\mathrm{H}^{n_1}(E(-n_1 + t - 1, t - 1))$. As in the case i), to construct a surjective map

$$\varphi : \mathrm{H}^0(E(t, t - 1)) \to \mathrm{H}^{n_1}(E(-n_1 + t - 1, t - 1))$$

we need to show $\mathrm{H}^i(E(t - i, t - 1)) = 0$ for $i = 1, \cdots, n_1$. In fact, we see $\mathrm{H}^i(E(t - i, t - 1)) = \mathrm{H}^i(E(-i + 1, 0) \otimes \mathcal{O}_X(t - 1, t - 1)) = 0$ for $i \neq n_1$ because $-i \leq (-i + 1) + 0 \leq 0$, $-n_1 \leq -i + 1 \leq 0$ and $-n_2 \leq 0 \leq 0$. Also, we see $\mathrm{H}^{n_1}(E(-n_1 + t, t - 1)) = 0$ by assumption. Thus there is a nonzero element $g \in \mathrm{H}^0(E(t, t - 1))$ such that $\varphi(g) = s(\neq 0) \in \mathrm{H}^{n_1}(E(-n_1 + t - 1, t - 1))$. Similarly, to construct a surjective map

$$\psi : \mathrm{H}^0(E^\vee(-t, -t + 1)) \to \mathrm{H}^{n_2}(E^\vee(-t, -t - n_2))$$

we need to show $\mathrm{H}^i(E(t - n_1 - 1, t + n_1 - 2 - i)) = 0$ for $i = n_1, \cdots, n_1 + n_2 - 1$. In fact, we see $\mathrm{H}^i(E(-n_1 + t - 1, n_1 + t - i - 2)) = \mathrm{H}^i(E(-n_1 + 1, n_1 - i) \otimes \mathcal{O}_X(t - 2, t - 2)) = 0$ for $i \neq n_1$ because $-i \leq (-n_1 + 1) + (n_1 - i) \leq 0$, $-n_1 \leq -n_1 + 1 \leq 0$ and $-n_2 \leq n_1 - i \leq 0$. Also, we see $\mathrm{H}^{n_1}(E(t - n_1 - 1, t - 2)) = 0$ by assumption. By taking a dual element $s^* \in \mathrm{H}^{n_2}(E^\vee(-t, -t - n_2))$ corresponding to $s \in \mathrm{H}^{n_1}(E(t - n_1 - 1, t - 1))$, we have a nonzero element $f \in \mathrm{H}^0(E^\vee(-t, -t + 1))$ such that $\psi(f) = s^*(\neq 0) \in \mathrm{H}^{n_2}(E^\vee(-t, -t - n_2))$. The elements g and f can be regarded as elements of $\mathrm{Hom}(\mathcal{O}_X(0, 1), E(t, t))$ and $\mathrm{Hom}(E(t, t), \mathcal{O}_X(0, 1))$ respectively. As in the case of i), we have that $f \circ g$ is an isomorphism. Hence $\mathcal{O}_X(0, 1)$ is a direct summand of $E(t, t)$.

Therefore the assertion is proved. □

Remark 2.4. Summing up the proof, we should note that an essential point is to use the assumption $\mathrm{H}^{n_1}(F(1, 0)) = \mathrm{H}^{n_1}(F(0, -1)) = 0$ for a vector bundle F with $\mathrm{H}^{n_1}(F) \neq 0$. Then F has a direct summand $\mathcal{O}_X(-n_1 - 1, 0)$.

In fact, we applied this procedure to the cases i) $F = E(-n_1 + t, t - 1)$, ii) $F = E(-n_1 + t - 1, t - 2)$ and iii) $F = E(-n_1 + t - 1, t - 1)$.

Finally we state a general result proved similarly as in Theorem 1.1.

Theorem 2.5. *Let E be a vector bundle on $X = \mathbb{P}_k^{n_1} \times \mathbb{P}_k^{n_2}$, where $n_1 \geq 1$ and $n_2 \geq 1$. Let r_1 and r_2 be integers such that $0 \leq r_1 \leq n_1$ and $0 \leq r_2 \leq n_2$. The vector bundle E is a direct sum of line bundles of $\mathcal{O}_X$, $\mathcal{O}_X(0,1), \cdots, \mathcal{O}_X(0, r_1)$, $\mathcal{O}_X(1,0), \cdots \mathcal{O}_X(r_2, 0)$ twisted by line bundles of the form $\mathcal{O}_X(\ell, \ell)$ if and only if*

$$\mathrm{H}^i(E(j_1 + t, j_2 + t)) = 0$$

for all integers i, j_1, j_2 and t satisfying that $1 \leq i \leq n_1 + n_2 - 1$, $-i \leq j_1 + j_2 \leq 0$, $-n_1 \leq j_1 \leq 0$ and $-n_2 \leq j_2 \leq 0$ except for either $i = n_1$ and $j_2 \geq j_1 + n_1 - r_1 + 1$, or $i = n_2$ and $j_2 \leq j_1 - n_2 + r_2 - 1$.

Outline of the Proof of Theorem 2.5. The "only if" part is easily shown by calculating cohomologies.

In order to show the "if" part we take the minimal integer t such that $E(t,t)$ is 0-regular. In case $\mathrm{H}^{n_1+n_2}(E(-n_1 + t - 1, -n_2 + t - 1) \neq 0$, we see that $\mathcal{O}_X$ is a direct summand of $E(t,t)$ as in the proof of Theorem 1.1. In particular, the case $r_1 = r_2 = 0$ is done. So we may assume that $\mathrm{H}^{n_1+n_2}(E(-n_1 + t - 1, -n_2 + t - 1) = 0$. From the assumption possible nonvanishing parts for the non-0-regularity of $E(t-1, t-1)$ appear in the n_1-th or n_2-th cohomologies. Thus we have only to consider the set of pairs (j_1, j_2) with $\mathrm{H}^i(E(j_1 + t, j_2 + t)) \neq 0$ for $i = n_1, n_2$.

Let us put $\mathfrak{S} = \{(j_1, j_2) | j_1 \geq -n_1 - 1, j_1 + n_1 - r_1 + 1 \leq j_2 \leq -j_1 - n_1 - 1\}$. Note that $\mathfrak{S}$ is nonempty if $r_1 \geq 1$, and we have $-n_1 \leq j_1 \leq -n_1/2 - 1$ and $-n_1 \leq j_2 \leq -1$ for $(j_1, j_2) \in \mathfrak{S}$. Since $E(t-1, t-1)$ is not 0-regular, we may assume there are some $(j_1, j_2) \in \mathfrak{S}$ such that $\mathrm{H}^{n_1}(E(j_1 + t, j_2 + t)) \neq 0$. If not, there are nonvanishing n_2-th cohomologies and then we can proceed similarly. For pairs (j_1, j_2) and (k_1, k_2) in $\mathfrak{S}$ we define a total order $(j_1, j_2) < (k_1, k_2)$ if either i) $j_2 - j_1 < k_2 - k_1$ or ii) $j_2 - j_1 = k_2 - k_1$ and $j_1 < k_1$. For example, in the case of (1.1), $\mathfrak{S}$ consists of $(-n_1 - 1, -2)$, $(-n_1 - 1, -1)$ and $(-n_1, -1)$ in the increasing order.

Now, back to the proof, we take the minimal $(j_1, j_2) \in \mathfrak{S}$ such that $\mathrm{H}^{n_1}(E(j_1 + t, j_2 + t)) \neq 0$. By taking $F = E(j_1 + t, j_2 + t)$, we have $\mathrm{H}^{n_1}(F) \neq 0$ and $\mathrm{H}^{n_1}(F(1,0)) = \mathrm{H}^{n_1}(F(0,-1)) = 0$ from the minimality of (j_1, j_2) in $\mathfrak{S}$ and the assumption. Then the corresponding proof of Theorem 1.1 works for $F = E(j_1 + t, j_2 + t)$ as described in Remark 2.4. Indeed, by using $\mathrm{H}^{n_1}(F(1,0)) = 0$, we have a surjective map $\varphi : \mathrm{H}^0(F(n_1 + 1, 0) \to \mathrm{H}^{n_1}(F)$. Similarly, by using $\mathrm{H}^{n_1}(F(0,-1)) = 0$, we have a surjective map $\psi : \mathrm{H}^0(F^\vee(-n_1 - 1, 0)) \to \mathrm{H}^{n_2}(F^\vee(-n_1 - 1, -n_2 - 1))$. Then there are elements $f \in \mathrm{H}^0(F(n_1 + 1, 0))$ and $g \in \mathrm{H}^0(F^\vee(-n_1 - 1, 0))$ such that $\varphi(f) \otimes \psi(g)$ gives a canonical element via $\mathrm{H}^{n_1}(F) \otimes \mathrm{H}^{n_2}(F^\vee(-n_1 - 1, -n_2 - 1)) \to \mathrm{H}^{n_1+n_2}(\mathcal{O}_X(-n_1 - 1, -n_2 - 1))$. Thus the corresponding maps $f : \mathcal{O}_X \to F(n_1 + 1, 0)$ and $g : F(n + 1, 0) \to \mathcal{O}_X$ satisfy that $g \circ f$ is an isomorphism.

Hence $\mathcal{O}_X(-j_1 - n_1 - 1, -j_2)$ is a direct summand of $E(t, t)$. Repeating this procedure, therefore, yields the assertion.

□

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