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Fibonacci Diffraction Grating

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A diffraction grating, which is called *Fibonacci diffraction grating* in this paper, is made by the Fibonacci sequence rule. A self-similar structure in a diffraction spectrum is observed experimentally when laser light is insident on the grating. In the diffraction spectrum, it is found that ratios of two successive intervals are equal to the golden mean, $(\sqrt{5}-1)/2$. This relationship also holds for sub-peaks. The wavelength of the incident light is also measured and found to be in good agreement with the accepted value.

§ 1. Introduction

In physics education experiments using diffraction gratings are widely performed to observe the diffraction phenomena. The main purpose of such experiments is to measure the wavelength of the incident light by using a conventional diffraction grating whose lines are drawn at equal intervals. We know that the diffraction spectrum is expressed by the power spectrum of the grating pattern since diffraction theory is closely related to Fourier transformations. Therefore, it is interesting educationally to observe various diffraction spectra from arbitrary grating patterns. If students can find some remarkable structures in spectrum, we expect that such an experiment will be more attractive.

Recently, in solid state physics it has been found that aperiodic or quasiperiodic structures in crystal lattices play an important role in expressing some properties of solid[1]. The Fibonacci lattice is one of the quasicrystals. It is known that its diffraction spectrum shows a fascinating structure characterized by self-similarity[1,2]. In the present paper we make a Fibonacci diffraction grating which is equivalent to the Fibonacci lattice and observe a diffraction spectrum experimentally. We aim in this paper to give an educational method in order to study diffraction theory and Fourier analysis through current topics.

§ 2. Theoretical basis

First, we briefly summarize diffraction theory for a conventional diffraction grating. The following discussion is for the case when light is incident perpendicular to the diffraction grating. We consider transmitted light in the direction of angle θ measured from the perpendicular axis. A schematic drawing is shown in Fig. 1, where d is equal to half of the lattice constant and E_j' s are the transmitted waves. When a transmitted wave E_0 can be expressed as

$$E_0 = A\exp[iks - i\omega t], \qquad (1)$$

where A, k and ω are the amplitude, the wavenumber and the angular frequency, respectively, E_j's

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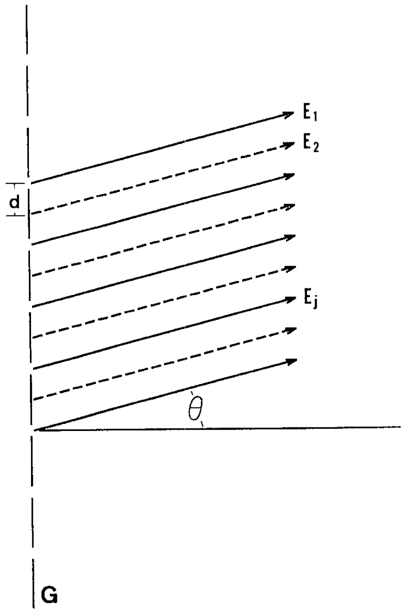

Fig. 1 A schematic drawing for the diffraction phenomenon. G is the diffraction grating and E_j's are the transmitted lights.

can be written as

$$\begin{aligned} E_1 &= 1\cdot A\exp[ik(s+\delta)-i\omega t], \\ E_2 &= 0\cdot A\exp[ik(s+2\delta)-i\omega t], \\ &\vdots \\ E_j &= x_j\cdot \exp[ik(s+j\delta)-i\omega t], \\ &\vdots \\ E_N &= x_N\cdot A\exp[ik(s+N\delta)-i\omega t] \end{aligned} \tag{2}$$

with

$$\delta = d\sin\theta. \tag{3}$$

For the conventional diffraction grating the space series $\{x_j\}$, which expresses the grating pattern, is described as

$$\{x_j\}=\{10101010\cdots\}. \tag{4}$$

Superimposing all E_j's, we obtain the expression of transmitted wave,

$$E=\sum_{j=1}^{N} x_j A\exp[ikj\delta]\cdot\exp[iks-i\omega t]. \tag{5}$$

The intensity of it is

$$I=|E|^2=|A\sum_j x_j\exp[ijk\delta]|^2. \tag{6}$$

Equation (6) is a power spectrum of space series $\{x_j\}$. For the conventional diffraction grating whose space series is expressed by Eq. (4), the diffraction spectrum, Eq. (6), can be written as the

δ-function with period π. Hence the following equation holds for all peaks with integer m:

$$k\delta = m\pi, \tag{7}$$

namely,

$$2d \sin\theta = m\lambda. \tag{8}$$

where λ is the wavelength.

Next, we introduce the Fibonacci sequence. Sequence $\{F_m\}$ made by the rule $F_m = F_{m-2} + F_{m-1}$ with seeds $F_0 = 1$ and $F_1 = 1$ is called the Fibonacci sequence. For the limit when $m \to \infty$, the ratio of two successive Fibonacci numbers is equal to the golden mean as is indicated by an equation

$$\lim_{m\to\infty} \frac{F_{m-1}}{F_m} = \rho = \frac{\sqrt{5}-1}{2}$$
$$\sim 0.618339\cdots. \tag{9}$$

The space series $\{x_j\}$ of the Fibonacci diffraction grating is made by the following rule[2]. First, we set $\{G_0\} = \{1\}$ and $\{G_1\} = \{0\}$ as two seeds. Next, we define a series $\{G_m\}$ constructed by symbols (numbers) 1 and 0 as

$$\{G_m\} = \{G_{m-2} G_{m-1}\}. \tag{10}$$

We show concretely G_m for $m \geq 2$ as the following:

$$\begin{aligned} \{G_2\} &= \{G_0 G_1\} = \{10\} \\ \{G_3\} &= \{G_1 G_2\} = \{010\} \\ \{G_4\} &= \{G_2 G_3\} = \{10010\} \\ \{G_5\} &= \{G_3 G_4\} = \{01010010\} \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \end{aligned}$$

The Fibonacci diffraction grating can be obtained by taking the series $\{G_m\}$ as the space series $\{x_j\}$,

$$\{x_j\} = \{G_m\} = \{1001001010010\cdots\}. \tag{11}$$

Namely, the grating pattern can be made by a rule which we draw a line for 1 and do not draw for 0 in Eq. (11). Figure 2 shows a pattern drawn by this rule.

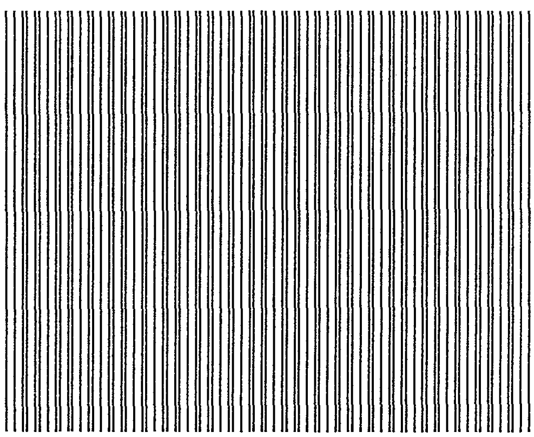

Fig. 2 A pattern of the Fibonacci diffraction grating.

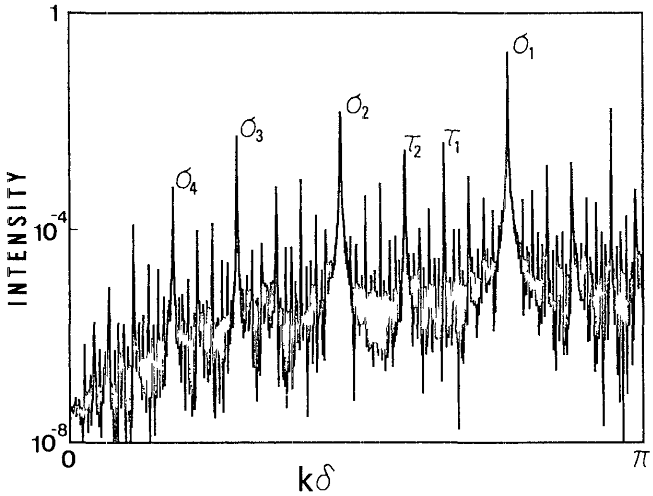

Fig. 3 A diffraction spectrum of the Fibonacci diffraction grating. Longitudinal axis is log scale and transverse axis is linear. Principal peaks and sub-peaks are indicated by σ_j's and τ_j's, respectively.

The diffraction spectrum can be calculated by Eq. (6) with Eq. (11). The result shown in Fig. 3 was calculated from Eq. (6) using the FFT method. We can see a line of principal peaks at positions σ_1, σ_2, $\cdots$ as they are indicated in Fig. 3. The relationship between consecutive peaks can be represented as

$$\frac{\sigma_2}{\sigma_1}=\frac{\sigma_3}{\sigma_2}=\cdots=\frac{\sigma_{m+1}}{\sigma_m}=\cdots=\rho \tag{12}$$

This relationship also holds true for sub-peaks. If we measure τ_j as the displacement from σ_k (the case of $k=2$ is shown in Fig. 3), a similar relation to Eq. (12) is satisfied among τ_j's. It is found from Eq. (12) that Fig. 3 shows a self-similar nested structure, namely fractal structure.

Furthermore, we determine the wavelength of the incident light from the relation

$$k\delta=\sigma_1=2\pi\rho^2 \tag{13}$$

We make an additional remark that the intensity I_p of principal peaks can be scaled by $I_p/(k\delta)^4$.

§ 3 Experiment and results

A photograph was taken of a drawing similar to Fig. 2 and the negative was used as the Fibonacci diffraction grating. The original drawing was made by a computer with space step 0.03 cm and entire length 18.3 cm. The lattice constant d, which is defined as the interval between symbols 1 and 0, was measured with a traveling microscope and we obtained its value as $d=1.28\times10^{-3}$ cm. The incident light we used was He-Ne laser whose wavelength λ is 6.328×10^{-5} cm. A typical diffraction image on a screen is shown in Fig. 4(a), where the distance between the diffraction grating and screen, D, was 196.5 cm. In order to show Fig. 4(a) more clearly, main spots are indicated again in Fig. 4(b).

Ratios of σ_j's, Eq. (12), were calculated from the experimental data which contains data σ_0's. We obtained $\rho_\sigma=0.625\pm0.005$. This value is very close to the true value ($\rho=0.618$). For

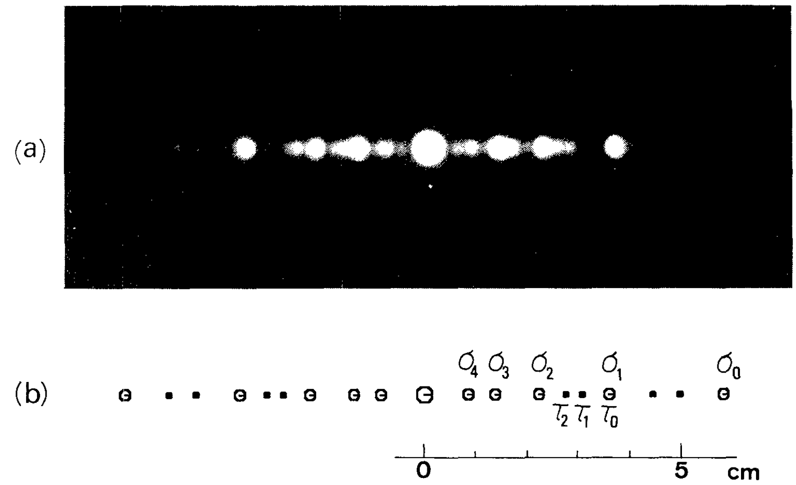

Fig. 4 (a) A photograph of the diffraction image on the screen.
(b) Main spots of Fig. 4(a). Principal peaks and sub-peaks are indicated by σ_j's and τ_j's, respectively. These are corresponding to the same indications in Fig. 3.

sub-peaks, τ_j's, $\rho_\tau=0.569\pm0.017$ is obtained by also using data τ_0's. Disagreement of the value ρ_τ with ρ is due to inaccuracy in the measurement of the position τ_2. It is difficult to distinguish the peak τ_2 from peaks near τ_2 because intensities of sub-peaks are very weak.

Let z_1 be the displacement of σ_1 from the center of the screen. The wavelength of the laser can be calculated from Eq. (13), which is rewritten as

$$\lambda=\frac{dz_1}{\rho^2 D} \tag{14}$$

by approximating $\sin\theta$ to z_1/D. The average value z_1 of left and right directions on the screen is 3.605 cm. From Eq. (14) $\lambda_{\mathrm{exp}}=6.15\times10^{-5}$ cm is obtained. This value is very close to λ of He-Ne laser.

§ 4. Conclusion

When we do this experiment on the Fibonacci diffraction grating, the diffraction spectrum can be observed in accordance with the theory. We obtain that the ratios of two successive intervals are coincident with the golden mean. The wavelength of the incident light can also be measured with sufficient precision.

We expect that the observation of the self-similar (fractal) structure in the diffraction spectrum is to stimulate students curiosity for optical studies. A similar power spectrum has also been found in a phenomenon of chaos[3]. The Fibonacci lattice or the phenomena characterized by the Fibonacci sequence become important in solid state physics and nonlinear physics. We consider that the present experiment is effective for students not only to study fundamental physics, such as diffraction theory and Fourier analysis, but also as an introduction into the current topics of nonlinear physics, such as chaos and fractals.

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