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# Perturbation Method for $\phi^4$-Field Equation

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The McLaughlin-Scott perturbation theory for soliton equation is applied to a $\phi^4$-field equation. A set of ordinary differential equations, which express time evolution of soliton parameters, is obtained. Inter-chain interracion is also considered.

**Key words :** soliton, perturbation theory, inter-chain interaction

## 1. Introduction

The $\phi^4$-field equation is frequently used as an equation which expresses structural phase transition. For example, it is well-known that this equation is a good model for describing dimerization in *trans*-polyacetylene.[1,2] The $\phi^4$-field equation is in the same class of the sine-Gordon equation. The unperturbed equation has a soliton solution and properties of soliton dynamics have been studied in many papers.[3]

In real systems effects of impurities and dissipation cannot be ignored. It is considered that soliton is trapped by impurities and locally oscillates when the external periodic force is applied. In a perturbed sine-Gordon equation system we have shown that such an oscillation bifurcates to chaos.[4–8] The McLaughlin-Scott perturbation theory is a useful method to study soliton dynamics under these situations.[9]

In this paper, we apply the McLaughlin-Scott perturbation theory to a perturbed $\phi^4$-field equation and derive a set of ordinary differential equations similar to the sine-Gordon equation system.

## 2. One-Soliton Solutions of $\phi^4$-Field Equation

The unperturbed $\phi^4$-field equation is described as

$$\phi_{tt} - \phi_{xx} + U'(\phi) = 0, \tag{1}$$

where

$$U(\phi) = \frac{1}{2}(\phi^2 - 1)^2 \tag{2}$$

and

$$U'(\phi) = 2(\phi^3 - \phi).$$

As is shown in Eq.(2), the potential $U(\phi)$ has double minima. In order to solve Eq.(1), we rewrite Eq.(1) as

$$(u^2 - 1)\frac{d^2\phi}{d\xi^2} + U'(\phi) = 0 \tag{3}$$

by using a variable transformation :

$$\xi = x - ut.$$

Multiplying $d\phi/d\xi$ to Eq.(3) and integrationg once, we obtain

$$\frac{1}{2}(u^2-1)\left(\frac{d\phi}{d\xi}\right)^2+U(\phi)=0, \tag{4}$$

where we choose an integral constant as 0. Further integration of Eq.(4),

$$\int\frac{d\phi}{1-\phi^2}=\pm\int\frac{d\xi}{\sqrt{1-u^2}},$$

yields

$$\eta=\pm\frac{\xi+x_0}{\sqrt{1-u^2}}$$

by putting $\phi=\tanh\eta$, where $x_0$ is an integral constant. As a result one-soliton solution can be written by

$$\phi(x,t)=\tanh\left[\pm\frac{x-ut+x_0}{\sqrt{1-u^2}}\right], \tag{5}$$

where $+$ denotes the kink solution and $-$ the anti-kink solution. The kink expresses the solution that the field $\phi$ varies from $-1$ to $+1$ and propagates with the velocity $u$. The soliton position $X$ is represented by $x_0-ut$.

## 3. Perturbation Method for $\phi^4$-Field Equation

We consider the following perturbed equation:

$$\phi_{tt}-\phi_{xx}+U'(\phi)=\epsilon f, \tag{6}$$

where $f$ is the perturbation described as

$$f=-\alpha\phi_t-\sum_j\mu_j\delta(x-q_j)U'(\phi)-\gamma(1+G\cos\Omega t). \tag{7}$$

Here the first term of the right hand side of Eq.(7) denotes the dissipation, the second term the inhomogeneous potential mede by impurities placed at $q_j$ and the third term the external force. Equation (6) can be rewritten by

$$\partial_t\boldsymbol{W}+\boldsymbol{N}(\boldsymbol{W})=\epsilon\boldsymbol{f}, \tag{8}$$

where

$$\boldsymbol{W}=\begin{pmatrix}\phi\\ \phi_t\end{pmatrix},$$

$$\boldsymbol{N}(\boldsymbol{W})=\begin{pmatrix}0 & -1\\ -\partial_{xx}+U'(\cdot) & 0\end{pmatrix}\begin{pmatrix}\phi\\ \phi_t\end{pmatrix}$$

and

$$\boldsymbol{f}=\begin{pmatrix}0\\ f\end{pmatrix}.$$

For one-kink state a vector $\boldsymbol{W}$ can be written by

$$\boldsymbol{W}=\begin{pmatrix}\tanh\left[\pm\frac{1}{\sqrt{1-u^2}}\left(x-\int_0^t udt-x_0\right)\right]\\ \mp\frac{u}{\sqrt{1-u^2}}\operatorname{sech}^2\left[\pm\frac{1}{\sqrt{1-u^2}}\left(x-\int_0^t udt-x_0\right)\right]\end{pmatrix}. \tag{9}$$

Time evolution of soliton parameters $p_k$ is expressed by an equation,

$$\frac{dW}{dt}=\sum_k \frac{\partial W}{\partial p_k}\frac{dp_k}{dt}. \tag{10}$$

In this case we choose as

$$p_1=x_0, \quad p_2=u. \tag{11}$$

When we introduce a vector $\boldsymbol{b}_j$ as

$$\boldsymbol{b}_1=J\frac{\partial \boldsymbol{W}}{\partial x_0}=\begin{pmatrix}\phi_{tx}\\ -\phi_x\end{pmatrix} \tag{12}$$

and

$$\boldsymbol{b}_2=J\frac{\partial \boldsymbol{W}}{\partial u}=\begin{pmatrix}-\phi_{tu}\\ \phi_u\end{pmatrix} \tag{13}$$

with

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

Eq.(8) can be written by a form of an inner product,

$$\left(\boldsymbol{b}_j,\sum_k \frac{\partial \boldsymbol{W}}{\partial p_k}\frac{dp_k}{dt}+\boldsymbol{N}(\boldsymbol{W})\right)=\epsilon(\boldsymbol{b}_j,\boldsymbol{f}). \tag{14}$$

where $j=1$, 2 and the inner product is defined as an integral,

$$(\boldsymbol{F},\boldsymbol{G})=\int \boldsymbol{F}^t(x)\boldsymbol{G}(x)dx.$$

Using relations,

$$\boldsymbol{b}_1{}^t\cdot\frac{\partial \boldsymbol{W}}{\partial x_0}=0$$

and

$$\boldsymbol{b}_2{}^t\cdot\frac{\partial \boldsymbol{W}}{\partial u}=0,$$

from Eq.(14) we obtain the following integrals.

$$\int dx[\phi_{tx}\phi_u-\phi_x\phi_{tu}]\frac{du}{dt}-\int dx[\phi_{tx}\phi_t+\phi_x(-\phi_{xx}+U'(\phi))]$$
$$=-\epsilon\int dxf\phi_x \tag{15}$$

and

$$\int dx[\phi_{tu}\phi_x-\phi_u\phi_{tx}]\frac{dx_0}{dt}-\int dx[\phi_{tu}\phi_t+\phi_u(-\phi_{xx}+U'(\phi))]$$
$$=-\epsilon\int dxf\phi_u \tag{16}$$

### 3.1 *Calculation of Eq.(15)*

The first term of the left hand side of Eq.(15) is rewitten by

$$L_{15}^{(1)}=\int dx[\phi_{tx}\phi_u-\phi_x\phi_{tu}]\frac{du}{dt}$$
$$=-\frac{1}{u}\int dx[\phi_{tt}\phi_u-\phi_t\phi_{tu}]\frac{du}{dt}$$
$$=\frac{1}{u}\int dx[\phi_t\phi_{tu}-\phi_{xx}\phi_u+U'(\phi)\phi_u]\frac{du}{dt}$$

because of

$$\phi_{tx}\sim-\frac{1}{u}\phi_{tt}$$

and

$$\phi_x \sim -\frac{1}{u}\phi_t.$$

Integrating by parts :

$$\int \phi_{xx}\phi_u dx = \phi_x\phi_u\Big|_{-\infty}^{\infty} - \int \phi_x\phi_{xu}dx,$$

we obtain

$$L_{15}^{(1)} = \frac{1}{u}\frac{d}{du}\left(\int dx\left[\frac{1}{2}\phi_t{}^2 + \frac{1}{2}\phi_x{}^2 + U(\phi)\right]\right)\frac{du}{dt}.$$

The integrand of this equation is the Hamiltonian density :

$$\mathcal{H}(\phi) = \frac{1}{2}\phi_t{}^2 + \frac{1}{2}\phi_x{}^2 + U(\phi).$$

Calculation of this integral is carried out as follows. From an integration for rest soliton,

$$\begin{aligned}
&\int dx\,\mathcal{H}_0(\phi) \\
&= \int dx \ \left[\frac{1}{2}\phi_x{}^2 + \frac{1}{2}(\phi^2-1)^2\right] \\
&= \int dx \ \left[\frac{1}{2}\,\mathrm{sech}^4(x-x_0) + \frac{1}{2}\,\mathrm{sech}^4(x-x_0)\right] \\
&= \frac{4}{3},
\end{aligned}$$

and Lorentz transformation, the Hamiltonian is calculated as

$$\int dx\,\mathcal{H}(\phi) = \frac{4}{3}\frac{1}{\sqrt{1-u^2}},$$

and hence we obtain

$$L_{15}^{(1)} = \frac{4}{3}\frac{u}{3(1-u^2)^{3/2}}. \tag{17}$$

The second term of the left hand side of Eq.(15) is equal to zero because the integrand is an odd function.

The right hand side of Eq.(15) can be rewritten as

$$-\epsilon\int dx f\phi_x = \epsilon\int dx f\phi_t.$$

The first term of this equation is calculated to be

$$\begin{aligned}
R_{15}^{(1)} &= -\alpha\frac{1}{u}\int \mathrm{d}x\phi_t{}^2 \\
&= -\alpha\frac{u}{1-u^2}\int dx\,\mathrm{sech}^4\frac{x-ut-x_0}{\sqrt{1-u^2}} \\
&= -\frac{4}{3}\alpha\frac{u}{\sqrt{1-u^2}}.
\end{aligned}$$

The second and third terms can be calculated as

$$\begin{aligned}
R_{15}^{(2)} &= -\sum_j \mu_j \frac{1}{u}\int dx\delta(x-q_j)\,U'(\phi)\phi t \\
&= -\sum_j \frac{\mu_j}{u}\left(\pm\frac{2u}{\sqrt{1-u^2}}\tanh\ \left[\pm\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\right]\ \mathrm{sech}^4\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\right) \\
&= -\frac{2}{\sqrt{1-u^2}}\sum_j \mu_j\tanh\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\,\mathrm{sech}^4\frac{q_j-ut-x_0}{\sqrt{1-u^2}}
\end{aligned}$$

and

$$R_{15}^{(3)} = -\frac{1}{u}\gamma(1+G\cos\Omega t)\int dx\phi_t$$
$$= \pm\frac{1}{\sqrt{1-u^2}}\gamma(1+G\cos\Omega t)\int \text{sech}^2\frac{x-ut-x_0}{\sqrt{1-u^2}}$$
$$= \pm 2\gamma(1+G\cos\Omega t)$$

From Eq.(17) and these results with respect to the right hand side, Eq.(15) can be described as

$$\frac{du}{dt} = -\alpha u(1-u^2) - \frac{3}{2}\sum_j \mu_j \tanh\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\text{sech}^4\frac{q_j-ut-x_0}{\sqrt{1-u^2}}$$
$$\pm\frac{3}{2}\gamma(1-u^2)^{3/2}(1+G\cos\Omega t), \tag{18}$$

where + denotes the kink and − the anti-kink.

3.2 *Calculation of Eq.(16)*

The first term of the left hand side of Eq.(16) is calculated as

$$\int dx[\phi_{tu}\phi_x - \phi_u\phi_{tx}]\frac{dx_0}{dt}$$
$$= \left(-\phi_u\phi_t\Big|_{-\infty}^{\infty} + \int dx[\phi_{tu}\phi_x + \phi_{ux}\phi_t]\right)\frac{dx_0}{dt}$$
$$= \int dx\frac{d}{du}(\phi_t\phi_x)\frac{dx_0}{dt}$$
$$= \frac{d}{du}\left(-\frac{u}{1-u^2}\int dx\,\text{sech}^4\frac{x-ut-x_0}{\sqrt{1-u^2}}\right)\frac{dx_0}{dt}$$
$$= -\frac{4}{3}\frac{d}{du}\left(\frac{u}{\sqrt{1-u^2}}\right)\frac{dx_0}{dt}$$
$$= -\frac{4}{3}\frac{1}{(1-u^2)^{3/2}}\frac{dx_0}{dt}.$$

The second term vanishes because the integrand is an odd function.

In the right hand side of Eq.(16), beacuse

$$\phi_u = \pm\frac{u(x-ut-x_0)}{(1-u^2)^{3/2}}\text{sech}^2\frac{x-ut-x_0}{\sqrt{1-u^2}}$$

is an odd function, the first and third terms become to be zero. The second term is calculated as

$$R_{16}^{(2)} = -\sum_j \mu_j\int dx\delta(q_j-x)U'(\phi)\phi_u$$
$$= \frac{2u}{(1-u^2)^{3/2}}\sum_j \mu_j(q_j-ut-x_0)\tanh\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\text{sech}^4\frac{q_j-ut-x_0}{\sqrt{1-u^2}}.$$

From these calculations Eq.(16) can be written by

$$\frac{dx_0}{dt} = -\frac{3}{2}u\sum_j \mu_j(q_j-ut-x_0)\tanh\frac{q_j-ut-x_0}{\sqrt{1-u^2}}\text{sech}^4\frac{q_j-ut-x_0}{\sqrt{1-u^2}}. \tag{19}$$

If we put

$$X = x_0 + ut, \tag{20}$$

we finally obtain a set of ordinary differential equations:

$$\frac{du}{dt}=-\alpha u(1-u^2)-\frac{3}{2}(1-u^2)\sum_j\mu_j\tanh\frac{q_j-X}{\sqrt{1-u^2}}\operatorname{sech}^4\frac{q_j-X}{\sqrt{1-u^2}}$$

$$\pm\frac{3}{2}\gamma(1-u^2)^{3/2}(1+G\cos\Omega t) \tag{21}$$

$$\frac{dX}{dt}=u-\frac{3}{2}u\sum_j\mu_j(q_j-X)\tanh\frac{q_j-X}{\sqrt{1-u^2}}\operatorname{sech}^4\frac{q_j-X}{\sqrt{1-u^2}} \tag{22}$$

For $q_j=0$ and $\mu_j=\mu$ this equation is simplified to be

$$\frac{du}{dt}=-\alpha u(1-u^2)+\frac{3}{2}(1-u^2)\tanh\frac{X}{\sqrt{1-u^2}}\operatorname{sech}^4\frac{X}{\sqrt{1-u^2}}$$

$$\pm\frac{3}{2}\gamma(1-u^2)^{3/2}(1+G\cos\Omega t) \tag{23}$$

$$\frac{dX}{dt}=u-\frac{3}{2}\mu uX\tanh\frac{X}{\sqrt{1-u^2}}\operatorname{sech}^4\frac{X}{\sqrt{1-u^2}} \tag{24}$$

## 4. Inter-chain Interaction

Now we consider a two-chain system, in which two solitons can couple with each other through inter-chain interaction. This situation can be described as a coupled form of $\phi^4$-field equation :

$$\phi^{(1)}_{tt}-\phi^{(1)}_{xx}+U'(\phi^{(1)})=-\alpha\phi^{(1)}-\sum_j\mu_j\delta(x-q_j)\quad U'(\phi^{(1)})-\gamma(1+G\cos\Omega t)+\varkappa(\phi^{(2)}-\phi^{(1)}), \tag{25}$$

$$\phi^{(2)}_{tt}-\phi^{(2)}_{xx}+U'(\phi^{(2)})=-\alpha\phi^{(2)}-\sum_j\mu_j\delta(x-q_j)\quad U'(\phi^{(2)})-\gamma(1+G\cos\Omega t)+\varkappa(\phi^{(1)}-\phi^{(2)}), \tag{26}$$

Here $\varkappa$ expresses the strength of coupling. Hereafter, we only consider one-kink solution,

$$\phi^{(i)}=\tanh\frac{x-u_i-x_i}{\sqrt{1-u_i^2}}=\tanh\Theta_i \tag{27}$$

For $u_1\sim u_2\sim 0$, we can approximate

$$\Theta_2\sim\Theta_1+X_1-X_2$$

An interaction term in the right hand side of Eq. (15) can be calculated as

$$R^{(I)}_{15}=\frac{\varkappa}{u}\int dx[\tanh(\Theta_1+X_1-X_2)-\tanh\Theta_1]\phi_t$$

$$=-\varkappa\int d\Theta[\tanh(\Theta_1+X_1-X_2)-\tanh\Theta_1]\operatorname{sech}^2\Theta$$

$$=-\varkappa\int d\Theta\frac{\tanh(X_1-X_2)}{1+\tanh\Theta\tanh(X_1-X_2)}\operatorname{sech}^4\Theta$$

$$=-\frac{4}{3}\varkappa\tanh(X_1-X_2).$$

Therefore, we obtain for interaction terms as

$$\left(\frac{du_1}{dt}\right)_I=\varkappa\tanh(X_2-X_1), \tag{28}$$

$$\left(\frac{du_2}{dt}\right)_I=\varkappa\tanh(X_1-X_2), \tag{29}$$

Similarly, for Eq. (16) because the interaction term is calculated as

$$R^{(I)}_{16}=\varkappa\int dx[\tanh(\Theta_1+X_1-X_2)-\tanh\Theta_1]\phi_u$$

$$= \varkappa u \int d\Theta [\tanh(\Theta_1 + X_1 - X_2) - \tanh\Theta_1] \Theta \mathrm{sech}^2\Theta$$
$$= \varkappa u \int d\Theta \frac{\tanh(X_1 - X_2)}{1 + \tanh\Theta \tanh(X_1 - X_2)} \Theta \mathrm{sech}^4\Theta$$
$$= 0,$$

then the interaction terms give no contribution :

$$\left(\frac{dx_1}{dt}\right)_I = 0, \quad \left(\frac{dx_2}{dt}\right)_I = 0.$$

Hence, the inter-cain interaction terms can be added to a set of ordinary differential equations such as

$$\frac{du_1}{dt} = -\alpha u_1(1-u_1^2) + \frac{3}{2}\mu(1-u_1^2)\tanh\frac{X_1}{\sqrt{1-u_1^2}}\mathrm{sech}^4\frac{X_1}{\sqrt{1-u_1^2}}$$
$$\pm\frac{3}{2}\gamma(1-u_1^2)^{3/2}(1+G\cos\Omega t) + \varkappa\tanh(X_2 - X_1) \quad (30)$$

$$\frac{dX_1}{dt} = u_1 - \frac{3}{2}\mu u_1 X_1 \tanh\frac{X_1}{\sqrt{1-u_1^2}}\mathrm{sech}^4\frac{X_1}{\sqrt{1-u_1^2}} \quad (31)$$

$$\frac{du_2}{dt} = -\alpha u_2(1-u_2^2) + \frac{3}{2}\mu(1-u_2^2)\tanh\frac{X_2}{\sqrt{1-u_2^2}}\mathrm{sech}^4\frac{X_2}{\sqrt{1-u_2^2}}$$
$$\pm\frac{3}{2}\gamma(1-u_2^2)^{3/2}(1+G\cos\Omega t) + \varkappa\tanh(X_1 - X_2) \quad (32)$$

$$\frac{dX_2}{dt} = u_2 - \frac{3}{2}\mu u_2 X_2 \tanh\frac{X_2}{\sqrt{1-u_2^2}}\mathrm{sech}^4\frac{X_2}{\sqrt{1-u_2^2}} \quad (33)$$

where we show the case in which one impurity is placed at $x=0$.

## 5. Conclusion

We show that a perturbed $\phi^4$-field equation which is a partial differential equation can be reduced to a set of ordinary differential equations by applying the McLaughlin-Scott perturbation theory. Time evolution of soliton position and velocity can be investigated by solving equations btained in the pesent paper. Moreover, when the inter-chain interaction exists, we can also show a reduced form by similar manner.

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