Conditions for the Steinness of a complex manifold

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Abstract

In this paper we give conditions for the Steinness of a complex space or a complex manifold X such that $\dim H^1(X, \mathcal{O}) < +\infty$.

0. Introduction

We may prove the following theorem using the method of J. E. Fornaess – R. Narasimhan [7] and B. Jennane [9], [10].

Let X be a second countable reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.

i) $H^1(X, \mathcal{O}) = 0$, *where $\mathcal{O}$ is the structure sheaf of X.*

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

In case that X is an unramified domain over a Stein manifold, the above statement remains valid even if we replace the condition i) by the weaker one that $\dim H^1(X, \mathcal{O}) < +\infty$. More precisely we have the following result.

Let X be a second countable complex manifold. Then X is a Stein manifold if the following three conditions are satisfied.

i) $\dim H^1(X, \mathcal{O}) < +\infty$, *where $\mathcal{O}$ is the structure sheaf of X.*

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

iii) *There exists a realization $\Phi : X \to S$ as a ramified domain over a Stein manifold S such that the ramification locus of it is an empty set or a Stein variety of X.*

In addition to the above two theorems we also give an improvement of a result in the author and M. Furushima [2].

1. Unramified domains over a Stein manifold

Throughout this paper all complex spaces are supposed to be second countable. We always denote by $\mathcal{O}$ the structure sheaf of the complex space X considered in each cases. A reduced closed complex subspace of a complex space X is called a variety of X.

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Let X and Z be complex spaces. Let $\Phi : X \to Z$ be a holomorphic map such that all fibers of Φ are discrete sets in X. Then $\Phi : X \to Z$ is called a ramified domain over Z. The unique minimal variety V of X such that the map $\Phi | (X - V)$ is locally biholomorphic is called the ramification locus of $\Phi : X \to Z$. If $V = \emptyset$, then $\Phi : X \to Z$ is called an unramified domain over Z.

LEMMA 1. *Let X be a reduced complex space such that* $\dim \mathrm{H}^1 (X, \mathcal{O}) < +\infty$. *Let f be a holomorphic function on X such that the set $Y = \{x \in X \mid f(x) = 0\}$ is a Stein variety of X. Then for every holomorphic function g on Y there exist an integer N and a holomorphic function G on X such that* $1 \leqq N \leqq \dim \mathrm{H}^1 (X, \mathcal{O}) + 1$ *and* $G | Y = g^N$.

PROOF. Y has a Stein neighbourhood U_0 by the main theorem of Y. - T. Siu [11]. We may assume that $g \in \mathcal{O}(U_0)$. $\mathscr{U} = \{U_0, U_1\}$ is an open covering of X, where $U_1 = X - Y$. $g/f \in \mathcal{O}(U_0 \cap U_1) = \mathrm{Z}^1(\mathscr{U}, \mathcal{O})$. Since the natural map $\mathrm{H}^1(\mathscr{U}, \mathcal{O}) \to \mathrm{H}^1(X, \mathcal{O})$ is injective, $\dim \mathrm{H}^1(\mathscr{U}, \mathcal{O}) \leqq \dim \mathrm{H}^1(X, \mathcal{O}) < +\infty$. Let N be a minimal integer such that $\{[(g/f)^k]\}_{k=1}^{N}$ is linearly dependent in $\mathrm{H}^1(U, \mathcal{O})$. Then $1 \leqq N \leqq \dim \mathrm{H}^1(X, \mathcal{O}) + 1$. There exist $c_1, \ldots, c_{N-1} \in \mathbf{C}$, $h_0 \in \mathcal{O}(U_0)$ and $h_1 \in \mathcal{O}(U_1)$ such that $\sum_{k=1}^{N-1} c_k (g/f)^k + (g/f)^N = h_1 - h_0$ on $U_0 \cap U_1 = U_0 - Y$. We can define $G \in \mathcal{O}(X)$ by the equations $G = \sum_{k=1}^{N-1} c_k g^k f^{N-k} + g^N + h_0 f^N$ on U_0 and $G = h_1 f^N$ on U_1. Then $G = g^N$ on Y.

LEMMA 2. *Let $\Phi : X \to S$ be an unramified domain over a Stein manifold S. Then X is a Stein manifold if the following two conditions are satisfied.*

i) $\dim \mathrm{H}^1(X, \mathcal{O}) < +\infty$.

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

PROOF. Without loss of generality we may assume that S is connected and of dimension n. In the proof we use the method of the proof of Theorem 7 of the author and Y. Abe [1]. Suppose that $\Phi : X \to S$ is not p_7- convex in the sense of F. Docquier and H. Grauert [6]. Then there exist H, P, φ and ψ with following properties. $H = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_j| < 1 (j = 1, \ldots, n)\} \cup \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid 1 - \varepsilon < |z_1| < 1 + \varepsilon, |z_j| < 1 + \varepsilon \ (j = 2, \ldots, n)\}$, $P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_j| < 1 + \varepsilon \ (j = 1, \ldots, n)\}$, $0 < \varepsilon < 1$. $\varphi : \bar{H} \to \widetilde{X} = X \cup \widetilde{\partial} X$ is a continuous map such that $\varphi(H) \subset X$. There exists $b = (b_1, \ldots, b_n)$ such that $|b_1| < 1 - \varepsilon$, $\max\limits_{2 \leqq j \leqq n}$

$|b_j| = 1$ and $\varphi(b) \in \widetilde{\partial} X$. Here $\widetilde{\partial} X$ denotes the ideal boundary of $\Phi : X \to S$ defined in F. Docquier and H. Grauert [6]. $\psi : P \to S$ is an injective holomorphic map such that $\Phi \circ (\varphi | H) = \psi | H$. Let $\lambda : X \to X'$, $\Phi' : X' \to S$ be an envelope of holomorphy of $\Phi : X \to S$. It holds that $\Phi' \circ (\lambda \circ (\varphi | H)) = \psi | H$. Since X' is a Stein manifold, there exists a holomorphic map $\varphi' : P \to X'$ such that $\varphi' | H = \lambda \circ (\varphi | H)$. $\varphi'(b)$ is a boundary point of $\lambda(X)$ in X'. There exists a holomorphic function f' on X' such that $f' \circ \lambda = f$. Put $t_0 = f'(\varphi'(b))$. Take an irreducible component A containing b of the variety $\{z \in P \mid (f' \circ \varphi')(z) = t_0\}$ of P. By the pseudoconvexity of $P - A$ it holds that $H \cap A \neq \emptyset$ Let $\gamma : [0,1] \to A$ such that $\gamma(0) \in H \cap A$ and $\gamma(1) = b$. Since $\varphi'(\gamma(0)) \in \lambda(X)$ and $\varphi'(\gamma(1)) = \varphi'(b) \notin \lambda(X)$, there exist a sequence $(q_\nu)_{\nu=1}^{\infty} \subset X$ and a boundary point p_0 of $\lambda(X)$ in X' such that $\lim_{\nu\to\infty} \lambda(q_\nu) = p_0$ and $\lambda(q_\nu) \in \varphi'(\gamma([0, 1]))$. Since $Y = \{x \in X | f(x) = t_0\}$ is Stein and $(q_\nu)_{\nu=1}^{\infty}$ is a divergent sequence in Y, there exists a holomorphic function g on Y which satisfies $\lim_{\nu\to\infty} |g(q_\nu)| = +\infty$. By Lemma 1 there exist an integer $N \geqq 1$ and a holomorphic function G on X such that $G | Y = g^N$. There exists a holomorphic function G' on X' such that $G' \circ \lambda = G$. It holds that $|G'(p_0)| = \lim_{\nu\to\infty} |g(q_\nu)|^N = +\infty$. It is a contradiction. Thus X is a Stein manifold by Satz 10 of F. Docquier and H. Grauert [6].

COROLLARY. *Let X be an open set of $\mathbf{C}^n$ such that* $\dim H^1(X, \mathcal{O}) < +\infty$. *Suppose that $X \cap \{z_1 = t\}$ is Stein for every $t \in \mathbf{C}$. Then X is a Stein manifold.*

2. Complex spaces with a vanishing cohomology group

LEMMA 3. *Lex X be a reduced complex space. Then X is holomorphically separable if the following two conditions are satisfied.*

i) $\dim H^1(X, \mathcal{O}) < +\infty$.

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

PROOF. Let x_0 and x_1 be two distinct points in X. In cace that $f(x_0) = f(x_1)$, we put $t_0 = f(x_0)$. $Y = \{x \in X \mid f(x) = t\}$ is Stein and $x_0, x_1 \in Y$. There exists a holomorphic function g on Y such that $g(x_0) = 0$ and $g(x_1) = 1$. By Lemma 1 there exist an integer $N \geqq 1$ and a holomorphic function G on X such that $G | Y = g^N$. $G(x_0) = g(x_0)^N = 0$, $G(x_1) = g(x_1)^N = 1$. It follows that X is holomorphically separable.

To prove Theorem 1 we use the following Lemmas 4 and 5 which is originally due to J. E. Fornaess and R. Narasimhan [7].

LEMMA 4. Let X be a reduced complex space of finite dimension such that $H^1(X,\mathcal{O}) = 0$. Let f be a holomorphic function on X such that the set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and an open set $U \Subset X$ such that $U \cap Y_t$ is a Runge Stein open set of Y_t and $K \cap Y_t \subset U \cap Y_t$ for $t \in \mathbf{C}$, $|t| < \varepsilon$.

PROOF. See the proof of Lemma 2.4 of J. E. Fornaess and R. Narasimhan [7]. Instead of Lemma 2.2 of [7] we use Lemma 1.

LEMMA 5. Let X be a reduced complex space of finite dimension without isolated points such that $H^1(X,\mathcal{O}) = 0$. Let f be a holomorphic function on X which is not constant on any irreducible component of X and assume that the set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and a compact set $L \subset X, K \subset L$, such that the following conditions are satisfied: for $t \in \mathbf{C}$, $|t| < \varepsilon$, there exists $M_t > 0$ such that for any $\tau > 0$ and for any holomorphic function g on Y_t there exists a holomorphic function G on X which satisfies $G \mid Y_t = g$ and $\| G \|_K \leqq M_t \| g \|_{L \cap Y_t} + \tau$.

PROOF. We can prove this lemma along the lines of the proof of Lemme 4 in B. Jennane [9].

THEOREM 1. *Let X be a reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.*

i) $H^1(X,\mathcal{O}) = 0$.

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

PROOF. In the proof we use the method of B. Jennane [10]. We first deal with the case that f is not constant on any irreducible component of X. Since X is holomorphically separable by Lemma 3, it suffices to prove that X is holomorphically convex. Let K be an arbitrary compact set in X and $s \in \mathbf{C}$. Put $Y_t = \{x \in X \mid f(x) = t\}$ for $t \in \mathbf{C}$. Then there exist $\varepsilon > 0$ and a compact set $L \subset X$ such that the statement of Lemma 5 holds for $f - s$ instead of f. Take $p \in Y_t - (L \cap Y_t)^{\wedge}_{Y_t}$ for $t \in \mathbf{C}$, $|t - s| < \varepsilon$. There exists a holomorphic function g on Y_t such that $|g(p)| > 1$ and $\| g \|_{L \cap Y_t} < 1/(2M_t)$. There exists $G \in \mathcal{O}(X)$ such that $G \mid Y_t = g$. $\| G \|_K \leqq M_t \| g \|_{L \cap Y_t} + 1/2 < 1$. Since $|G(p)| = |g(p)| > 1$, $p \notin \hat{K}_X$. Therefore $\hat{K}_X \cap Y_t \subset (L \cap Y_t)^{\wedge}_{Y_t}$ for $t \in \mathbf{C}$, $|t - s| < \varepsilon$. By Lemma 4 there exists $\delta(s) > 0$ such that $\delta(s) < \varepsilon$ and $\bigcup_{|t-s| < \delta(s)}$

$(L \cap Y_t)^{\wedge}_{Y_t} \Subset X$. Then we have $\hat{K}_X \cap \{x \in X \mid |f(x)-s| < \delta(s)\} = \bigcup_{|t-s|<\delta(s)} (\hat{K}_X \cap Y_t) \Subset X$. There exists finitely many $s_1, \cdots, s_N \in \mathbf{C}$ such that $\{t \in \mathbf{C} \mid |t| \leqq \| f \|_K\} \subset \bigcup_{j=1}^{N} \{t \in \mathbf{C} \mid |t-s_j| < \delta(s_j)\}$. Since $\hat{K}_X \subset \{x \in X \mid |f(x)| \leqq \| f \|_K\}$, $\hat{K}_X = \bigcup_{j=1}^{N} (\hat{K}_X \cap \{x \in X \mid |f(x)-s_j| < \delta(s_j)\}) \Subset X$. It follows that X is holomorphically convex.

We now deal with the general case. Let X' be the union of irreducible components of X on which f is constant and X'' be the union of those on which f is not constant. Since every irreducible component of X' is a variety of the Stein space $\{x \in X \mid f(x) = c\}$ for some $c \in \mathbf{C}$, X' is a Stein variety of X. Let $\mathcal{J}'$ and $\mathcal{J}''$ be the defining ideals of the varieties X' and X'' respectively. We have the exact sequence $0 \to \mathcal{O} \xrightarrow{\rho} (\mathcal{O}/\mathcal{J}') \oplus (\mathcal{O}/\mathcal{J}'') \xrightarrow{\tau} \mathcal{O}/(\mathcal{J}'+\mathcal{J}'')$, where ρ is the homomorphim $h_x \to (h_x + \mathcal{J}'_x,\ h_x + \mathcal{J}''_x)$ and τ is the homomorphism $(g_x + \mathcal{J}'_x,\ h_x + \mathcal{J}''_x) \to g_x - h_x + (\mathcal{J}'_x + \mathcal{J}''_x)$. From this we deduce the exact sequence of cohomology groups $\mathrm{H}^1(X, \mathcal{O}) \to \mathrm{H}^1(X', \mathcal{O}_X) \oplus \mathrm{H}^1(X'', \mathcal{O}_{X''}) \to \mathrm{H}^1(X, \mathcal{O}/(\mathcal{J}'+\mathcal{J}''))$, where $\mathcal{O}_{X'} = (\mathcal{O}/\mathcal{J}') \mid X'$ and $\mathcal{O}_{X''} = (\mathcal{O}/\mathcal{J}'') \mid X''$. $\operatorname{supp}(\mathcal{O}/(\mathcal{J}'+\mathcal{J}'')) = X' \cap X'' \subset X'$. Since X' is Stein, $\mathrm{H}^1(X', \mathcal{O}_{X'}) = 0$ and $\mathrm{H}^1(X, \mathcal{O}/(\mathcal{J}'+\mathcal{J}'')) = \mathrm{H}^1(X', (\mathcal{O}/(\mathcal{J}'+\mathcal{J}'')) \mid X') = 0$. $\mathrm{H}^1(X, \mathcal{O}) = 0$. Therefore $\mathrm{H}^1(X'', \mathcal{O}_{X''}) = 0$. Then by what was shown above X'' is Stein. It follows $X = X' \cup X''$ is Stein.

Using Theorem 1 we obtain the following result which is an improvement of Theorem 8 of the author and M.Furushima [2].

THEOREM 2. *Let X be a K–complete reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.*

i) $\dim \mathrm{H}^1(X, \mathcal{O}) < +\infty$.

ii) *For every holomorphic function f on X which is not constant on any positive dimensional irreducible component of X, the set $\{x \in X \mid f(x) = 0\}$ is a Stein variety of X.*

PROOF. Without loss of generality we may assume that X has no isolated point. Let F be the set of all holomorphic functions on X which are not constant on any irreducible component of X. Since X is K–complete, $F \neq \emptyset$. Every $f \in F$ satisfies the condition ii) of Theorem 1, so it suffices to prove that $\mathrm{H}^1(X, \mathcal{O}) = 0$ using the method of E. Ballico [5]. Let $f \in F$. Since the set $Y = \{x \in X \mid f(x) = 0\}$ is nowhere dense in X, the multiplication by f defines the monomorphism $m(f): \mathcal{O} \to \mathcal{O}$. The exact sequence $0 \to \mathcal{O} \xrightarrow{m(f)} \mathcal{O} \to \mathcal{O}/f\mathcal{O} \to 0$ induces the exact sequence of cohomology groups $\mathrm{H}^1(X, \mathcal{O})$

$\overset{m(f)^*}{\to} \mathrm{H}^1(X,\mathcal{O}) \to \mathrm{H}^1(X,\mathcal{O}/f\mathcal{O})$. Since the complex space $(Y,(\mathcal{O}/f\mathcal{O})\mid Y)$ is Stein by a theorem of H. Grauert (Theorem 5 of [8], p. 154), $\mathrm{H}^1(X,\mathcal{O}/f\mathcal{O}) = \mathrm{H}^1(Y,(\mathcal{O}/f\mathcal{O})\mid Y) = 0$. Therefore $m(f)^*$ is surjective. Since $\mathrm{H}^1(X,\mathcal{O})$ is of finite dimension, $m(f)^*$ is an isomorphism. Suppose that $\mathrm{H}^1(X,\mathcal{O}) \neq 0$ and take $\xi \in \mathrm{H}^1(X,\mathcal{O})$, $\xi \neq 0$. There exists $(a_1,\ldots,a_N) \in \mathbf{C}^N - \{0\}$ such that $\sum_{j=1}^{N} a_j m(f^j)^*(\xi) = 0$, where $N = \dim \mathrm{H}^1(X,\mathcal{O}) + 1$. Since $\sum_{j=1}^{N} a_j f^j \in F$, $\sum_{j=1}^{N} a_j m(f^j)^*(\xi) = m(\sum_{j=1}^{N} a_j f^j)^*(\xi) \neq 0$. It is a contradiction. It follows that $\mathrm{H}^1(X,\mathcal{O}) = 0$.

Theorem 1 was obtained implicitly by B. Jennane [10] in the proof of his main result. Using the method of E. Ballico [5] we reduced Theorem 2 to Theorem 1. But in the category of K−complete reduced complex spaces the condition ii) in Theorem 2 is stronger than the condition ii) in Theorem 2.

3. Remified domains over a Stein manifold

We give a condition for the Steinness of a complex manifold which includes Lemma 2 as a special case.

THEOREM 3. *Let X be a complex manifold. Then X is a Stein manifold if the following three conditions are satisfied.*

i) $\dim \mathrm{H}^1(X,\mathcal{O}) < +\infty$.

ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$.*

iii) *There exists a realization $\Phi: X \to S$ as a ramified domain over a Stein manifold S such that the ramification locus of it is an empty set or a Stein variety of X.*

PROOF. Without loss of generality we may assume that X is connected. Let V be the ramification locus of $\Phi: X \to S$. V is a variety of X of pure codimension 1. Since V is Stein, there exists a Stein open set U_0 which includes V by the main theorem of Y.−T. Siu [11]. $\mathscr{U} = \{U_0, U_1\}$ is an open covering of X, where $U_1 = X - V$. By Satz 1 of F. Docquier and H. Grauert [6], $U_0 \cap U_1 = U_0 - V$ is a Stein manifold. From the Mayer−Vietoris exact sequence $\cdots \to \mathrm{H}^1(X,\mathcal{O}) \to \mathrm{H}^1(U_0,\mathcal{O}) \oplus \mathrm{H}^1(U_1,\mathcal{O}) \to \mathrm{H}^1(U_0 \cap U_1,\mathcal{O}) \to \cdots$ we have a surjection $\mathrm{H}^1(X,\mathcal{O}) \to \mathrm{H}^1(U_1,\mathcal{O})$. $\dim \mathrm{H}^1(U_1,\mathcal{O}) \leqq \dim \mathrm{H}^1(X,\mathcal{O}) < +\infty$. $\Phi \mid U_1: U_1 \to S$ is an unramified domain over S. The set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbf{C}$. $Y_t \cap U_1 = \{x \in U_1 \mid f(x) = t\} = Y_t - V$. $Y_t \cap U_0$ is a Stein neighbourhood of the boundary of an open set $Y_t \cap U_1$ in the reduced Stein space Y_t. $(Y_t \cap U_0) \cap (Y_t \cap U_1) = Y_t \cap (U_0 - V)$ is Stein. Therefore $Y_t \cap U_1$ is a Stein open set of Y_t by Theorem 4 of A. Andreotti and R. Narasimhan [4]. It follows that U_1 is Stein by Lemma 2. From the Mayer-Vietoris exact sequence $\cdots \to H^{i-1}(U_0 \cap U_1, \mathcal{O}) \to H^i(X, \mathcal{O}) \to H^i(U_0, \mathcal{O}) \oplus H^i(U_1, \mathcal{O}) \to \cdots$ we have that $H^i(X, \mathcal{O}) = 0$ for $i \geqq 2$. Then $H^1(X, \mathcal{O}) = 0$ by the theorem of L. Alessandrini [3] and Lemma 3. It follows that X is a Stein manifold by Theorem 1.

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