ON THE NEBENHÜLLE OF AN OPEN SET IN A STEIN MANIFOLD

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Abstract

Let D be a relatively compact open set in a Stein manifold. We prove that $(N(D), D)$ is a Runge pair if and only if D has Property (N). Such an open set D has disk property.

0. Introduction.

In this paper we revise and generalize the results in the author's [1]. Let D be a relatively compact open set in a Stein manifold X. We denote by $N(D)$ the schlicht Nebenhülle of D. In section 1 we prove that the following two conditions are equivalent: i) $(N(D), D)$ is a Runge pair, ii) for any compact set K in D there exists a Stein open set D' in X such that $\hat{K}_{D'} \subset D \Subset D'$. In section 2 we prove that such an open set D that $(N(D), D)$ is a Runge pair has disk property. The converse of this fact does not hold.

1. Property (N).

Let X be a Stein manifold. For every open set D in X we denote by $\mathscr{N}_D$ the family of all Stein open sets in X containing $\bar{D}$. We call the set $N(D) :=(\bigcap_{F\in\mathscr{N}_D} F)^{\circ}$ the *(schlicht) Nebenhülle* of D. $N(D)$ is a Stein open set in X containing D. If $\mathring{\bar{D}} = D$ and $\bar{D}$ has a Stein neighbourhood basis, then it holds that $N(D) = D$.

S. Sato [9] asserts that $\bar{D}$ has a Stein neighbourhood basis if D is a bounded domain in $\mathbf{C}^N$ such that $N(D) = D$ (Proposition 1 of [9]). But his assertion is not true. According to B. Stensønes [10] there exists a domain D in $\mathbf{C}^2$ such that $N(D) = D$ and $\bar{D}$ does not have a Stein neighbourhood basis.

PROPOSITON 1. *If D is a relatively compact open set in a Stein manifold X, then $N(N(D)) = N(D)$.*

PROOF. Since we can prove that $\mathscr{N}_{N(D)} = \mathscr{N}_D$, it holds that $N(N(D)) = N(D)$.

Let D' and D'' be open sets in a complex manifold X such that D' has a countable topology and $D'' \subset D'$. (D', D'') is called a *Runge pair* if the image of the restriction map $\mathscr{O}(D') \to \mathscr{O}(D'')$ is dense in $\mathscr{O}(D'')$. As is well-known (D', D'') is a Runge pair of Stein open sets if and only if $\hat{K}_{D''} = \hat{K}_{D'}$ for any compact set K in D''.

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PROPOSITON 2. *The following two conditions are equivalent for any open set D in a complex manifold X.*

i) *For any compact set $K \subset D$ there exists a Stein open set D' in X such that $\bar{D} \subset D'$ and $\hat{K}_{D'} \subset D$.*

ii) *For any compact set $K \subset D$ there exists a Runge pair (D', D'') of Stein open sets in X such that $\bar{D} \subset D'$ and $K \subset D'' \Subset D$.*

PROOF. i) → ii). We can find an analytic polyhedron D'' in D' such that $\hat{K}_{D'} \subset D'' \Subset D$. Then (D', D'') is a Runge pair (see [7], p. 295).

ii) → i). Since (D', D'') is a Runge pair of Stein open sets, $\hat{K}_{D'} = \hat{K}_{D''} \subset D'' \subset D$.

As in [1] we say that an open set D in a complex manifold X has *Property* (N) if one of the equivalent conditions in Proposition 2 is satisfied. Such an open set is necessarily a Stein open set. Moreover we have the following proposition.

PROPOSITION 3. *Let D be an open set in a complex manifold X. If D has Property* (N), *then for any compact set $K \subset D$ it holds that $\hat{K}_D = \{ x \in \bar{D} \mid |f(x)| \leqq \|f\|_K$ for all $f \in \mathcal{O}(\bar{D}) \}$. Especially such an open set D is convex with respect to $\mathcal{O}(\bar{D})$.*

PROOF. There exists a Runge Pair (D', D'') of Stein open sets such that $\bar{D} \subset D'$ and $K \subset D'' \Subset D$. It holds that $\hat{K}_{D''} = \hat{K}_D = \hat{K}_{D'}$. Let $L := \{ x \in \bar{D} \mid |f(x)| \leqq \|f\|_K$ for all $f \in \mathcal{O}(\bar{D}) \}$. Since $\mathcal{O}(D') \subset \mathcal{O}(\bar{D})$, $L \subset \hat{K}_{D'}$. Since $\mathcal{O}(\bar{D}) \subset \mathcal{O}(D'')$, $\hat{K}_{D''} \subset L$. It follows that $L = \hat{K}_D$.

LEMMA 4. *Let D be a relatively compact open set in a Stein manifold X. If $N(D) = D$, then D has Property* (N).

PROOF. By Remmert's embedding theorem [8] we may assume that X is a closed complex submanifold of some $\mathbf{C}^N$. Then by a theorem of F. Docquier and H. Grauert [4] there exist a Stein neighbourhood V of X and a holomorphic retraction $\rho : V \to X$ (see [5], p.257). Let K be an arbitrary compact set in D. Let $\varepsilon := d(K, \mathbf{C}^N - \rho^{-1}(D))$, then $\varepsilon > 0$. Here we denote by d the ordinary Euclidean distance of $\mathbf{C}^N$. Since $L := \bigcap_{F \in N_D} F$ is a compact set in X, there exist finitely many points $p_1, p_2, \ldots, p_\ell \in L - D$ such that $L - D \subset \bigcup_{j=1}^{\ell} (X \cap B(p_j, \varepsilon/2))$, where $B(p_j, \varepsilon/2) = \{ x \in \mathbf{C}^N \mid d(x, p_j) < \varepsilon/2 \}$.
Since the interior of L in X equals to D, there exists $q_j \in X \cap B(p_j, \varepsilon/2) - L$ for each j. $E := D \cup (\bigcup_{j=1}^{\ell} (X \cap B(p_j, \varepsilon/2))) - \{q_1, q_2, \ldots, q_\ell\}$ is a relatively compact open set in X containing L. For every $r \in \partial E$ there exists $F_r \in \mathcal{N}_D$ such that $r \notin F_r$. There exists a Stein open set F_r' in X such that $\bar{D} \subset F_r' \Subset F_r$. Since ∂E is compact, there

exist finitely many $r_1, r_2, \ldots, r_s \in \partial E$ such that $\partial E \subset \bigcup_{j=1}^{s} (X - \bar{F}_{r_j}')$. Then $G := (\bigcap_{j=1}^{s} F_{r_j}') \cap E$ is a Stein open set containing $\bar{D}$. Suppose that there exist $p \in \hat{K}_G - D$. Since $p \in E - D$, it holds that $d(p, q_j) < d(p, p_j) + d(p_j, q_j) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for some j. $\rho(q_j) = q_j \in G$. Thus $d(p, \mathbf{C}^N - \rho^{-1}(G)) < \varepsilon$. We can prove that $\rho^{-1}(G)$ is a Stein open set in $\mathbf{C}^N$ (see the proof Theorem 4 of J. Kajiwara [6]). Therefore $d(\hat{K}_{\rho^{-1}(G)}, \mathbf{C}^N - \rho^{-1}(G)) = d(K, \mathbf{C}^N - \rho^{-1}(G)) \geqq \varepsilon$. Since $\hat{K}_G \subset \hat{K}_{\rho^{-1}(G)}$, it holds that $d(\hat{K}_G, \mathbf{C}^N - \rho^{-1}(G)) \geqq d(\hat{K}_{\rho^{-1}(G)}, \mathbf{C}^N - \rho^{-1}(G)) \geqq \varepsilon$.

Thus $d(p, \mathbf{C}^N - \rho^{-1}(G)) \geqq d(\hat{K}_G, \mathbf{C}^N - \rho^{-1}(G)) \geqq \varepsilon$. It is a contradiction. It follows that $\hat{K}_G \subset D$.

The converse of the above lemma does not hold. For example consider the bounded domain $D := \{ t \in \mathbf{C} \mid |t| < 1 \} - [0, 1)$ in $\mathbf{C}$. It holds that $N(D) = \{ t \in \mathbf{C} \mid |t| < 1\} \neq D$. Since D is simply connected, the image of the restriction map $\mathcal{O}(\mathbf{C}) \to \mathcal{O}(D)$ is dense in $\mathcal{O}(D)$ by Runge's theorem. Therefore for any compact set K in D, it holds that $\hat{K}_{\mathbf{C}} \subset D$. It follows that D has Property (N).

Here we prove the following theorem which is a generalization of Lemma 4.

THEOREM 5. *The following two conditions are equivalent for any relatively compact open set D in a Stein manifold X.*

i) *$(N(D), D)$ is a Runge pair.*

ii) *D has Property* (N).

PROOF. i) → ii). Let K be an arbitrary compact set in D. By assumption $\hat{K}_D = \hat{K}_{N(D)}$. By Proposition 1 and Lemma 4 $N(D)$ has property (N). There exists a Runge pair (D', D'') of Stein open sets such that $K \subset D'' \Subset N(D) \Subset D'$. Then we have that $\hat{K}_{D''} = \hat{K}_{N(D)} = \hat{K}_{D'}$. It follows that $\hat{K}_{D'} = \hat{K}_D \subset D \Subset D'$.

ii) → i). Let K be an arbitrary compact set in D. By assumption there exists a Runge pair (D', D'') of Stein open sets such that $K \subset D'' \Subset D \Subset D'$. Then we have that $\hat{K}_{D''} = \hat{K}_D = \hat{K}_{D'}$. Since $D \subset N(D) \subset D'$, it holds that $\hat{K}_D \subset \hat{K}_{N(D)} \subset \hat{K}_{D'}$. Therefore $\hat{K}_D = \hat{K}_{N(D)}$. It follows that $(N(D), D)$ is a Runge pair of Stein open sets.

2. Disk property.

Let $\Delta := \{ t \in \mathbf{C} \mid |t| < 1 \}$. Let X be a complex manifold and D an open set in X. As in [1] we say that D has *disk property* if it satisfies the condition that if $\psi : \bar{\Delta} \to X$ is a continuous map holomorphic in Δ such that $\psi(\bar{\Delta}) \subset \bar{D}$ and $\psi(\partial \Delta) \subset D$, then $\psi(\bar{\Delta}) \subset D$. If X is a Stein manifold and D has disk property, then we can prove that D is p_7 – convex in the sense of F. Docquier and H. Grauert [4], therefore D is a Stein open set. The converse is not true. For example the set $\{(t_1, t_2, \ldots, t_N) \in \mathbf{C}^N \,||\, t_1| < |t_2| < \ldots < |t_N| < 1\}$ is a Stein open set in $\mathbf{C}^N$ which does not have disk property (see [1], p. 184). We remark that if D is a locally Stein open set with C^1 –smooth boundary in a complex manifold X, then D has disk property. We can prove this fact using the Bremermann's continuity theorem (schlicht version of Proposition 3 of the author's [2]).

PPOPOSITION 6. *Let D be an open set in a complex manifold X. If D has Property* (N), *then D has disk property.*

PROOF. Let $\psi : \bar{\Delta} \to X$ be a continuous map holomorphic in Δ such that $\psi(\bar{\Delta}) \subset \bar{D}$ and $\psi(\partial\Delta) \subset D$. Since $\psi(\partial\Delta) \subset D$ and ψ is continuous, there exists a number r such that $0 < r < 1$ and $\psi(\{t \in \mathbf{C} \mid r \leqq |t| \leqq 1\}) \subset D$. $\psi(\{t \in \mathbf{C} \mid |t| = r\})$ is a compact set contained in D. By hypothesis there exists a Stein open set D' such that $\bar{D} \subset D'$ and $\psi(\{t \in \mathbf{C} \mid |t| = r\})\hat{}_{D'} \subset D$. $\psi|_\Delta : \Delta \to D'$ is a holomorphic map. $\psi(\{t \in \mathbf{C} \mid |t| \leqq r\}) = \psi(\{t \in \mathbf{C} \mid |t| = r\}\hat{}_\Delta) \subset \psi(\{t \in \mathbf{C} \mid |t| = r\})\hat{}_D \subset D$ (Lemma 3 of [1]). Thus we obtain that $\psi(\Delta) \subset D$.

THEOREM 7. *Let D be a relatively compact open set in a Stein manifold X. If (N(D), D) is a Runge pair, then D has disk property.*

PROOF. By Theorem 5 and Proposition 6.

The converse of this theorem does not hold. We can prove that the bounded domain Ω_r in K. Diederich and J. E. Fornaess [3] does not have Property (N) if $r > \exp(2\pi)$. Therefore ($N(\Omega_r)$, Ω_r) is not a Runge pair if $r > \exp(2\pi)$. But Ω_r has disk property, since it is a Stein domain in $\mathbf{C}^2$ with C^∞ –smooth boundary.

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