Characterization of pseudoconvexity through the disk property for open sets with C^1 smooth boundaries

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Dedicated to Professor Joji Kajiwara on his sixtieth birthday

ABSTRACT. Let X be a complex manifold and D an open set with C^1-smooth boundary in X. Then D is pseudoconvex if and only if D has the disk property.

0. Introduction.

Let X be a complex manifold and D be an open set in X. Let $\Delta := \{t \in \mathbf{C} \mid |t| < 1\}$. As in [1] or [3] we say that D has the *disk property* if it satisfies the condition that if $\varphi : \overline{\Delta} \to X$ is a continuous map holomorphic in Δ such that $\varphi(\overline{\Delta}) \subset \overline{D}$ and $\varphi(\partial\Delta) \subset D$, then $\varphi(\overline{\Delta}) \subset D$. If the open set D has the disk property, then D is pseudoconvex. But the converse is not true. For example the n-dimensional Hartogs triangle $\{(t_1, t_2, \ldots, t_n) \in \mathbf{C}^n \mid |t_1| < |t_2| < \cdots < |t_n| < 1\}$ is a Stein open set in $\mathbf{C}^n$ which does not have the disk property ([1], p.184). The author [3] proved that if D is a relatively compact open set in a Stein manifold X such that $(N(D), D)$ is a Runge pair, then D has the disk property. Here we denote by $N(D)$ the schlicht Nebenhülle of D. The author [3] also remarked without proof that if D is a pseudoconvex open set with C^1 boundary in a complex manifold X, then D has the disk property. In this paper we give a proof of this fact. More precisely we have the following theorem.

Let X be a complex manifold and D an open set with C^1-smooth boundary in X. Then the following three conditions are equivalent.

1) *D is pseudoconvex.*

2) *D has the disk property.*

3) *If $\varphi : \Delta \to X$ is a holomorphic map such that $\varphi(\Delta) \subset \overline{D}$ and $\varphi(\Delta) \cap D \neq \emptyset$, then $\varphi(\Delta) \subset D$.*

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* 一般科目（数学）

平成5年9月1日受付

1. Preliminaries.

Let X be a complex manifold and D an open set in X. D is said to be *pseudoconvex* or *locally Stein* if for every $p \in \partial D$ there exists an open neighbourhood U such that $D \cap U$ is Stein. D is said to be an open set *with C^1-smooth boundary* if for every $p \in \partial D$ there exist an open neighbourhood U and a function $h : U \to \mathbf{R}$ of class C^1 such that $D \cap U = \{x \in U \mid h(x) < 0\}$ and $dh \neq 0$ on U. We already gave the definition of the disk property in the introduction.

LEMMA 1. *Let X be a complex manifold and D be an open set in X. If D has the disk property, then D is pseudoconvex.*

PROOF. Take an arbitrary point $p \in \partial D$. Let U be a connected Stein open neighbourhood of p. Suppose that $D \cap U$ were not Stein. Then $D \cap U$ is not p_7-convex in U in the sense of Docquier-Grauert [5]. Let $n := \dim U$. There exist ε, g and $(b_1, \ldots, b_n)$ which satisfy the following properties. $P := \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid |t_j| < 1 + \varepsilon \ (j = 1, \ldots, n)\}$, $Q := \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid |t_j| < 1 \ (j = 1, \ldots, n)\} \cup \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid 1 - \varepsilon < |t_1| < 1 + \varepsilon, \ |t_j| < 1 + \varepsilon \ (j = 2, \ldots, n)\}$, $0 < \varepsilon < 1$. $g : P \to U$ is an injective holomorphic map such that $g(Q) \subset D \cap U$. $g(b_1, \ldots, b_n) \notin D \cap U$, $|b_1| < 1 - \varepsilon$ and $\max_{2 \leq j \leq n} |b_j| = 1$. Then $\varphi := g(\cdot, b_2, \ldots, b_n) : \{t \in \mathbf{C} \mid |t| < 1 + \varepsilon\} \to X$ is a holomorphic map such that $\varphi(\overline{\Delta}) \subset \overline{D}$, $\varphi(\partial \Delta) \subset D$ and $\varphi(b_1) \in \partial D$. It is a contradiction. Therefore $D \cap U$ is Stein. It follows that D is pseudoconvex. $\square$

For the proof of our theorem we need the following continuity theorem which is a generalization of 2.2 of Bremermann [4]. It is also the schlicht version of Proposition 3 of the author's [2].

LEMMA 2. *Let D be a Stein open set in $\mathbf{C}^n$. Let U be an open neighbourhood of the interval $[0, 1]$ in $\mathbf{C}$ and $f : \Delta \times U \to \mathbf{C}^n$ a holomorphic map. Let $f_t := f(\cdot, t) : \Delta \to \mathbf{C}^n$ for $t \in [0, 1]$. If $f_t(\Delta) \subset D$ for every $t \in [0, 1)$, then it holds that $f_1(\Delta) \subset \partial D$ or $f_1(\Delta) \subset D$.*

PROOF. Suppose the assertion false. Then there exist α, $\beta \in \Delta$ such that $f_1(\alpha) \in D$ and $f_1(\beta) \in \partial D$. The linear fractional function $l : \Delta \to \Delta$, $l(w) := (w - \alpha)/(\overline{\alpha} w - 1)$, is biholomorphic. The map $g : \Delta \times U \to \mathbf{C}^n$, $g(w, t) := f(l(w), t)$, is holomorphic. Since $\Delta \times U$ and D are Stein, $g^{-1}(D)$ is also Stein. We consider the distance $\delta(w, t)$ from the point $(w, t) \in g^{-1}(D)$ to the boundary $\partial(g^{-1}(D))$ along the $(1, 0)$-direction. The function $-\log \delta$ is plurisubharmonic on $g^{-1}(D)$ (Theorem 4.22 of Hitotumatu [6], p. 74). The set $W := \{t \in \mathbf{C} \mid (0, t) \in g^{-1}(D)\}$ is an open neighbourhood of the interval $[0, 1]$ in

U. Let $R(t) := \delta(0,t)$ for $t \in W$. The function $-\log R$ is subharmonic on W. Since $g(\Delta \times [0,1)) = f(\Delta \times [0,1)) \subset D$, $\Delta \times \{t\} \subset g^{-1}(D) \subset \Delta \times U$ for every $t \in [0,1)$. Therefore $R(t) = 1$ for every $t \in [0,1)$. By Proposition préliminaire of Oka [8], p. 20, $-\log R(1) = \limsup_{t \to 1-0}(-\log R(t)) = 0$. Therefore $R(1) = 1$. Since $g(l^{-1}(\beta),1) = f_1(\beta) \in \partial D$, $(l^{-1}(\beta),1) \in g^{-1}(D)$. Therefore $R(1) \leq |l^{-1}(\beta)| < 1$. It is a contradiction. □

2. Theorem.

THEOREM. *Let X be a complex manifold and D be an open set with C^1-smooth boundary in X. Then the following three conditions are equivalent.*

1) *D is pseudoconvex.*

2) *D has the disk property.*

3) *If $\varphi : \Delta \to X$ is a holomorphic map such that $\varphi(\Delta) \subset \overline{D}$ and $\varphi(\Delta) \cap D \neq \emptyset$, then $\varphi(\Delta) \subset D$.*

PROOF. 1) → 3). We use the idea of the proof of Proposition 2.1 of Kerzman-Rosay [7], which asserts that a bounded pseudoconvex open set in $\mathbf{C}^n$ with C^1-smooth boundary is taut. Let $\varphi : \Delta \to X$ be a holomorphic map such that $\varphi(\Delta) \subset \overline{\Delta}$ and $\varphi(\Delta) \cap D \neq \emptyset$. Suppose that $\varphi(\Delta) \not\subset D$. Then there exists a boundary point c of $\varphi^{-1}(\partial D)$ in Δ. $p := \varphi(c) \in \partial D$. $n := \dim_p X$. There exists a holomorphic chart $(V; z_1, \cdots, z_n)$ such that $p \in V$ and that $D \cap V$ is Stein. We may regard V as an open set of $\mathbf{C}^n$. Let T be the unit inner normal vector of $\partial(D \cap V)$ at p. There exist an open set W and $\varepsilon_0 > 0$ such that $p \in W \subset\subset V$ and that $x + \varepsilon T \in D \cap V$ if $x \in \overline{D} \cap \overline{W}$ and $0 < \varepsilon \leq \varepsilon_0$. There exists an open neighbourhood U of the interval $[0, \varepsilon_0]$ in $\mathbf{C}$ such that $x + \lambda T \in V$ if $x \in \overline{D} \cap \overline{W}$ and $\lambda \in U$. There exists $\alpha > 0$ such that $P := \{t \in \mathbf{C} \mid |t - c| < \alpha\} \subset\subset \Delta$ and $\varphi(P) \subset W$. The map $f : P \times U \to V$, $f(t,\lambda) := \varphi(t) + \lambda T$, is holomorphic. Let $f_\lambda := f(\cdot, \lambda) : P \to V$ for every $\lambda \in [0, \varepsilon_0]$. $f_\lambda(P) \subset D \cap V$ for every $\lambda \in (0, \varepsilon_0]$. Since $D \cap V$ is a Stein open set in $\mathbf{C}^n$, it holds that $f_0(P) \subset D \cap V$ or $f_0(P) \subset \partial(D \cap V)$ by Lemma 2. $f_0(c) = p \in \partial(D \cap V)$. On the other hand $f_0(P) \cap D = \varphi(P) \cap D \neq \emptyset$ since $c \in \partial(\varphi^{-1}(\partial D))$. It is a contradiction.

3) → 2). Clear.

2) → 1). By Lemma 1. □

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