A remark on the 1-convex open covering

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Abstract. We give a theorem on the 1-convex open covering of order ≤ 2 which is a generalization of the theorem of Cho-Shon on the finite simple chain Stein open covering.

Keywords: Stein space, 1-convex space.

0. Introduction

If D_1 and D_2 are two Stein open sets in a complex space S, then the union $D_1 \cup D_2$ is not necessarily Stein. Tovar [8] proved that if X is a union of two relatively compact Stein open sets D_1 and D_2 in a reduced Stein space S such that $\dim H^1(X, \mathcal{O}_X) < +\infty$ then X is also a Stein open set in S (Theorem 3 of [8] or Theorem 1.1 of [3]). As a generalization of it Cho-Shon [3] obtained the following result (Proposition 3.4 of [3]).

Let X be an open set of a finite dimensional holomorphically separable complex space S. Assume that S is Stein or that $X \subset\subset S$. Assume that $\dim H^1(X, \mathcal{O}_X) < +\infty$ and that there exists a finite simple chain Stein open covering $\{D_i\}_{i=1}^N$ of X. Then the union $\cup_{i=1}^k D_i$ is Stein for every $k = 1, 2, \ldots, N$.

In this paper we give a theorem on the 1-convex open covering of order ≤ 2 which is a generalization of this result of Cho-Shon [3]. It is also a slight improvement of Theorem 9 of the author's [1].

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1. Preliminaries

Throughout this paper all complex spaces are supposed to be second countable. Let X be a (not necessarily reduced) complex space. A compact analytic set C in X is said to be the *maximal compact analytic set* of X if every nowhere discrete compact analytic set of X is contained in C and $\dim_x C > 0$ for every $x \in C$ (Grauert [4], p. 339). A complex

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space X is said to be 1-*convex* if X is holomorphically convex and contains the maximal compact analytic set. If the open sets D_1 and D_2 in a complex space are 1-convex, then the intersection $D_1 \cap D_2$ is also 1-convex.

Let X be a complex space and L a compact set of X. X is said to be *K-separable* outside L if for every $x \in X - L$ the analytic set $\{y \in X \mid [f](y) = [f](x) \text{ for every } f \in \mathcal{O}_X(X)\}$ is of dimension 0. A complex space X is 1-convex if and only if X is holomorphically convex and K-separable outside a compact set. A complex space X is K-complete if and only if X is K-separable outside the empty set $\emptyset$ ([6], p. 226).

An open covering $\{D_i\}_{i\in I}$ of a complex space X is said to be *of order* ≤ 2 if for all pairwise different three indices i_0, i_1 and i_2 the intersections $D_{i_0} \cap D_{i_1} \cap D_{i_2}$ are empty ([7], p. 18). If a finite open covering $\{D_i\}_{i=1}^N$ of a complex space X is a finite simple chain covering of X in the sense of Definition 3.3 of Cho-Shon [3], then it is of order ≤ 2.

2. Theorem

We give the following theorem which is a generalization of Proposition 3.4 of Cho-Shon [3]. In the proof we use Theorem 9 of the author's [1].

THEOREM. *Let X be a complex space which is K-separable outside a compact set. Assume that the dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite and that there exists a locally finite 1-convex open covering $\{D_i\}_{i\in I}$ of X of order ≤ 2. Then for every $J \subset I$ the union $\cup_{i\in J} D_i$ is 1-convex if it is K-separable outside a compact set.*

PROOF. $K := I - J$. $Y := \cup_{j\in J} D_j$. $Z := \cup_{k\in K} D_k$. $Y \cup Z = X$. $Y \cap Z = \cup_{(j,k)\in J\times K}(D_j \cap D_k)$ (disjoint union). By Theorem 9 of the author's [1] we have only to prove that $\dim H^1(Y, \mathcal{O}_X) < +\infty$. We have the Mayer-Vietoris exact sequence $\cdots \to H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_X) \oplus H^1(Z, \mathcal{O}_X) \to H^1(Y \cap Z, \mathcal{O}_X) \to \cdots$. It holds that $\dim H^1(X, \mathcal{O}_X) < +\infty$ by Siu's theorem (Proposizione 7 of Ballico [2] or Théorème 2 of Jennane [5]). $H^1(Y\cap Z, \mathcal{O}_X) = \prod_{(j,k)\in J\times K} H^1(D_j \cap D_k, \mathcal{O}_X)$. Since $D_j \cap D_k$ is 1-convex, it holds that $\dim H^1(D_j \cap D_k, \mathcal{O}_X) < +\infty$ by Narasimhan's theorem (Lemma 2 of the author's [1]). By assumption there exists a compact set L such that X is K-separable outside L. $I' := \{i \in I \mid D_i \cap L \neq \emptyset\}$. If $j \in J - I'$ or $k \in K - I'$, then we have that $H^1(D_j \cap D_k, \mathcal{O}_X) = 0$ since $D_j \cap D_k$ is Stein. Since the set I' is finite, it holds that $\dim H^1(Y \cap Z, \mathcal{O}_X) < +\infty$. Thus we obtain that $\dim H^1(Y, \mathcal{O}_X) < +\infty$ from the exact sequence above. □

Since every open set of a K-complete complex space is K-complete, we have the following corollary on the Stein open covering of order ≤ 2.

COROLLARY. *Let X be a K-complete complex space. Assume that the dimension of $H^1(X, \mathcal{O}_X)$ is at most countably infinite and that there exists a locally finite Stein open covering $\{D_i\}_{i\in I}$ of X of order ≤ 2. Then for every $J \subset I$ the union $\cup_{i\in J} D_i$ is Stein.*

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