# On the recurrent and transient sets of some time-inhomogeneous diffusion processes

**Yoichi Oshima (Kumamoto University)**

## 1. Preliminaries

**Example 1** $\mathbf{M} = (X_t, P_x)$ :diffusion process with generator

$$\mathcal{G}\varphi(x) = \frac{1}{2}\frac{d^2\varphi}{dx^2} + \frac{d\log\rho_0}{dx}\frac{d\varphi}{dx}$$

Associated Dirichlet form on $L^2(X;m)$ ($dm = \rho_0^2(x)dx$) :

$$\mathcal{E}(\varphi,\psi) = \frac{1}{2}\int_X \frac{d\varphi}{dx}\frac{d\psi}{dx}\rho_0^2(x)dx.$$

Take $\rho(t,x)$ such that $\rho(0,x) = \rho_0(x)$ and put

$$\mathcal{G}^{(t)}\varphi(x) \equiv \frac{1}{2}\frac{d^2\varphi}{dx^2} + \frac{\partial\log\rho}{\partial x}\frac{d\varphi}{dx}$$

Associated Dirichlet form $\mathcal{E}^{(t)}$ on $d\mu_t(dx) = \rho^2(t,x)dx$ :

$$\mathcal{E}^{(t)}(\varphi,\psi) = \frac{1}{2}\int_X \frac{d\varphi}{dx}\frac{d\psi}{dx}\rho^2(t,x)dx$$

$\mathbf{M}^\rho$ : associated time inhomogeneous diffusion:
i.e. $p^\rho(s,x;t,\varphi) \equiv E^\rho_{(s,x)}\left(\varphi(X_{t-s})\right)$ satisfies

$$\frac{\partial}{\partial s}p^\rho(s,x;t,\varphi) = -\frac{1}{2\rho^2(s,x)}\frac{\partial}{\partial x}\left(\rho^2(s,x)\frac{\partial}{\partial x}p^\rho(s,x;t,\varphi)\right)$$

with $p^\rho(t,x;t,\varphi) = \varphi(x)$

$\widehat{p}^\rho(t,y;s,\psi) = \widehat{E}^\rho_{(t,y)}\left(\psi(\widehat{X}_{t-s})\right)$: transition function of dual process $\widehat{\mathbf{M}}^\rho$ of $\mathbf{M}^\rho$:

$$\frac{\partial}{\partial t}\left(\rho^2(t,y)\widehat{p}^\rho(t,y;s,\psi)\right) = \frac{1}{2}\frac{\partial}{\partial y}\left(\rho^2(t,y)\frac{\partial}{\partial y}\widehat{p}^\rho(t,y;s,\psi)\right)$$

with initial condition $\widehat{p}(s, y; s, \psi) = \psi(y)$

**Problem** To give conditions on $\rho$ for the recurrent or transience of some sets relative to $\mathbf{M}^\rho$

**General framework**:

$X$: locally compact separable metric space
$m$: a positive Radon measure on $X$, $\mathrm{Supp}[m] = X$.

$(\mathcal{E}, \mathcal{F})$: strongly local Dirichlet form on $L^2(X; m)$

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \int_X d\mu_{\langle \varphi, \psi \rangle}(x), \quad \varphi, \psi \in \mathcal{F}.$$

$\mathbf{M} = (X_t, P_x)$: diffusion process corresponding to $(\mathcal{E}, \mathcal{F})$.

$\mathcal{G}$: generator of $\mathbf{M}$
Fix $\rho(t, x) > 0$ such that

$$\rho(t, \cdot) \in \mathcal{F}, \quad \frac{\partial \rho}{\partial t}(t, \cdot) \in L^2(X; m)$$

Put

$$\begin{aligned} d\mu_t(x) &= \rho^2(t, x) dm(x) \\ \mathcal{E}^{(t)}(\varphi, \psi) &= \frac{1}{2} \int_X \rho^2(t, x) d\mu_{\langle \varphi, \psi \rangle}(x) \end{aligned}$$

$\mathbf{M}^\rho = (X_t, P^\rho_{(s,x)})$: time-inhomogeneous diffusion process corresponding to $\mathcal{E}^{(t)}$ on $L^2(d\mu_t)$:

$p^\rho(s, x; t, dy)$: transition function of $\mathbf{M}^\rho$

$R^\rho_\alpha \varphi(s, x) = \int_0^\infty e^{-\alpha t} p^\rho(s, x; s + t, dy) \varphi(y)$ :resolvent

# 2. Condition for recurrence of a set

$F$: fixed open set $D \equiv X \setminus F$.
Normalize $\mu_t$ as $\mu_t(D) = 1$ $\forall t$.
Take $f \geq 0$ such that $\int_D f(x) d\mu_s(x) = 1$ ($s$: fix) and

$$\begin{aligned} \widehat{u}_s(t, y) &= \widehat{E}^\rho_{(t,y)} \left( f(\widehat{X}_{t-s}) : t - s < \widehat{\sigma}_F \right) \\ \widehat{H}_s(t) &= \int_D \widehat{u}^2_s(t, y) d\mu_t(y) \\ \lambda_D(t) &= - \inf_{x \in D} \frac{\partial}{\partial t} \log \rho^2(t, x) \end{aligned}$$

Then $\widehat{u}_s(t,y)$ is a solution of

$$\frac{1}{\rho^2(t,y)}\frac{\partial}{\partial t}\left(\rho^2\widehat{u}_s\right)=\widehat{\mathcal{G}}^{(t)}\widehat{u}_s$$

Multiplying $\widehat{u}_s(t,y)$ and integrating by $\mu_t$,

**Lemma 2**

(i) $\mathcal{E}^{(t)}\left(\widehat{u}_s(t,\cdot),\widehat{u}_s(t,\cdot)\right)\le -\frac{1}{2}\frac{d\widehat{H}_s}{dt}+\frac{1}{2}\lambda_D(t)\widehat{H}_s(t)$

(ii) If $\lim_{t\to\infty}\widehat{H}_s(t)=0$, then $P^{\rho}_{f\cdot\mu_s}(\sigma_F<\infty)=1$

In (i) of Lemma 2,

$$\left(\inf_{x\in D}\rho^2(t,x)\right)\mathcal{E}\left(\widehat{u}_s(t,\cdot),\widehat{u}_s(t,\cdot)\right)\le\mathcal{E}^{(t)}\left(\widehat{u}_s(t,\cdot),\widehat{u}_s(t,\cdot)\right)$$

Also, by Fitzsimmons

$$\int_D\varphi^2\rho^2(t,x)dm(x)\le 2\|R^D\rho^2(t,\cdot)\|_\infty\mathcal{E}(\varphi,\varphi) \tag{1}$$

($R^D$ : potential of the part process of $\mathbf{M}$ on $D$). Put

$$\delta_D(t)=\frac{\inf_{x\in D}\rho^2(t,x)}{\|R^D\rho^2(t,\cdot)\|_\infty}$$

Then

$$\frac{1}{2}\delta_D(t)\widehat{H}(t)\le-\frac{1}{2}\frac{d\widehat{H}}{dt}+\frac{1}{2}\lambda_D(t)\widehat{H}(t).$$

**Lemma 3** For any $T>s$,

$$\widehat{H}_s(T)\le\|f\|^2_{\mu_s}\exp\left(\int_s^T\left(\lambda_D(t)-\delta_D(t)\right)dt\right).$$

**Theorem 4** If

$$\lim_{T\to\infty}\int_s^T\left(\lambda_D(t)-\delta_D(t)\right)dt=-\infty$$

then $P_{f\cdot\mu_s}\left(\sigma_F<\infty\right)=1$.

**Remark** If $q_D\equiv\|R^D1\|_\infty<\infty$, then

$$\delta_D(t)\ge q_D^{-1}\frac{\inf_{x\in D}\rho^2(t,x)}{\sup_{x\in D}\rho^2(t,x)}$$

**Example 5** For a locally bounded non-negative function $U\in\mathcal{F}$, let

$$D_{a,b}=\{x:a<U(x)<b\}$$

Suppose

$$\rho^2(t,x) = e^{-\beta(t)U(x)}/Y(t), \quad \beta(t) = \frac{1}{c}\log(1+t)$$

for

$$Y(t) = \int_{D_{a,b}} e^{-\beta(t)U(x)} dm(x)$$

Then

$$\begin{aligned}\lambda_{a,b}(t) &= -\inf_{D_{a,b}} \frac{\partial}{\partial t}\log\rho^2(t,x) = \sup_{x\in D_{a,b}} \frac{U(x)}{c}\frac{1}{1+t} + (\log Y(t))' \\ &= \frac{b}{c}\,\frac{1}{1+t} + (\log Y(t))'\end{aligned}$$

If $\|R^{D_{a,b}}1\|_\infty \equiv q_{D_{a,b}} < \infty$,

$$\begin{aligned}\inf_{D_{a,b}}\rho^2(t,\cdot) &= \frac{1}{Y(t)}(1+t)^{-b/c} \quad \sup_{D_{a,b}}\rho^2(t,\cdot) = \frac{1}{Y(t)}(1+t)^{-a/c} \\ \delta_{D_{a,b}} &\geq q_{D_{a,b}}^{-1}(1+t)^{-(b-a)/c}\end{aligned}$$

Hence Theorem 4 implies

$$b-a<c \quad \Rightarrow \quad P^\rho_{f\cdot\mu_s}(\sigma_{X\setminus D_{a,b}}<\infty) = 1$$

# 3. Conditions for transience for a set

For $\Gamma = (0,T)\times F$, put

$$u^T_{\alpha,\beta}(s,x) = E^\rho_{(s,x)}\left(\int_0^\infty e^{-\alpha t-\beta A_t}\beta dA_t\right)$$

where $A_t = \int_0^t I_\Gamma(s+\tau, X_\tau)d\tau$.
Then, if $g\leq 0$ on $F$,

$$-\left(\frac{\partial u^T_{\alpha,\beta}}{\partial t}, g\rho^2(t,\cdot)\right)_m + \mathcal{E}^{(t)}_\alpha(u^T_{\alpha.\beta}, g) = \Big((1-u^T_{\alpha,\beta})I_\Gamma, g\rho^2(t,\cdot)\Big)_m \leq 0$$

Put $g = u^T_{\alpha,\beta} - v$ for a function $v=1$ on $\Gamma$

**Assumption (A.1)** $\left|\frac{\partial}{\partial t}\rho^2\right| \leq k_0$ for some constant $k_0$.

Under (A.1),

$$\begin{aligned}&\left(\frac{\alpha-k_0}{\alpha}\right)^2 \int \mathcal{E}^{(t)}_\alpha\left(u^T_{\alpha,\beta}(t,\cdot), u^T_{\alpha,\beta}(t,\cdot)\right)dt \qquad (2)\\ \leq\ & \frac{2}{\alpha}\int\int\left(\rho^{-1}\frac{\partial(v\rho^2)}{\partial t}\right)^2 dmdt + 2\int\mathcal{E}_\alpha\left(v(t,\cdot),v(t,\cdot)\right)dt\end{aligned}$$

**Assumption (A.2)** There exists $v(t,x) = \xi(t)\psi(x)$ such that $0 \le \xi, \psi \le 1$, $\xi(t) = 1$ $(t \ge 0)$, $\psi = 1$ on $F$ and

$$\int\int \left(\rho^{-1}\frac{\partial(v\rho^2)}{\partial t}\right)^2 dmdt + \int \mathcal{E}_\alpha\left(v(t,\cdot), v(t,\cdot)\right) dt < \infty$$

Under (A.2), by letting $\beta \to \infty$ and $T \to \infty$ in (2)

$$h_\alpha(t,x) = E^\rho_{(t,x)}\left(e^{-\alpha\sigma_F}\right) = \lim_{T\to\infty}\lim_{\beta\to\infty} u^T_{\alpha,\beta}(t,x)$$

satisfies (2). In particular, $\|h_\alpha(t,\cdot)\|_{L^2(\mu_s)} \to 0$ $(t \to \infty)$

**Theorem 6** Under (A.1) and (A.2),

$$\lim_{s\to\infty} P_{\mu_s}\left(\sigma_F < \infty\right) = 0$$

**Case of Example 5**

In Example 5, suppose that $m(X) < \infty$ and $U$ is bounded. Let $D = D_b$ be a connected component of $\{x : U(x) < b\}$ and $F = X \setminus D_b$, $a = \inf_{x\in D_b} U(x)$.
Since $\log\rho^2 = -(U(x)/c)\log(1+t) - \log Y(t)$, (A.1) holds.
If $b - a > c$, then by taking $v$ such that $Supp[v] \subset \{x; U(x) > b - \epsilon\}$, it holds that

$$\left(\rho^{-1}\frac{\partial(v\rho^2)}{\partial t}\right)^2 \le k(1+t)^{-(b-\epsilon-a)/c}$$

Hence (A.2) holds. Thus

If $b - a > c$, then $P_{\mu_s}(\sigma_F < \infty) \to 0$, $(s \to \infty)$.
Combining this with the result stated in Example 5, we can say the followings:

$\mathbf{M}^\rho_{b,1}$ : process corresponding to $(b - c - \epsilon_1) \vee U(x) \wedge b$
$\Rightarrow$ $\mathbf{M}^\rho_{b,1}$ does not hits $\{x : U(x) > b\}$ approximately.

$\mathbf{M}^\rho_{b,2}$ : process corresponding to $(b - c + \epsilon_2) \vee U(x) \wedge b$
$\Rightarrow$ $\mathbf{M}^\rho_{b,2}$ exits $D = \{x : b - c + \epsilon_2 \le U(x) \le b\}$ a.s.

Since $\mathbf{M}^\rho_{b,1}$ and $\mathbf{M}^\rho_{b,2}$ are same until they exit from
$\{x : b - c + \epsilon_2 \le U(x) \le b\}$, $\mathbf{M}^\rho$ approximately hit the set $\{x : U(x) < b - c + \epsilon_2\}$ before $\{x : U(x) \ge b\}$ for any $\varepsilon > 0$.

# 4. Limit behavior

Let

$$\begin{aligned}
\widehat{u}_s^{(0)}(t,y) &= \widehat{E}^{\rho}_{(t,y)}\left(f(\widehat{X}_{t-s})\right) \\
\lambda^{(0)}(t) &= -\inf_{y\in X}\frac{\partial}{\partial t}\log\rho^2(t,y)
\end{aligned}$$

Then

$$\mathcal{E}^{(t)}\left(\widehat{u}_s^{(0)},\widehat{u}_s^{(0)}\right) \le -\frac{1}{2}\frac{d\widehat{H}^{(0)}}{dt} + \frac{1}{2}\lambda^{(0)}(t)\widehat{H}^{(0)} + 2\left\|\frac{\partial\rho}{\partial t}\right\|_m \sqrt{\widehat{H}^{(0)}} \tag{3}$$

**Assumption (A.3)** There exists $\eta(t)$ (spectral gap) such that

$$\eta(t)\int_X \left(v(x) - \langle\mu_t, v\rangle\right)^2 \rho^2(t,x)dm(x) \le \mathcal{E}^{(t)}(v,v), \quad v\in\mathcal{F} \tag{4}$$

Then

$$\eta(t)\widehat{H}^{(0)}(t) \le -\frac{1}{2}\frac{d\widehat{H}^{(0)}}{dt} + \frac{1}{2}\lambda^{(0)}(t)\widehat{H}^{(0)}(t) + 2\left\|\frac{\partial\rho}{\partial t}\right\|_{L^2(m)}\sqrt{\widehat{H}^{(0)}}$$

**Lemma 7** Under (A.3),

$$\begin{aligned}
\widehat{H}^{(0)}(T)^{1/2} &\le \exp\left(\frac{1}{2}\int_s^T\left(\lambda^{(0)}(t) - 2\eta(t)\right)dt\right) \\
\times &\quad \left(\widehat{H}^{(0)}(s)^{1/2} + 2\int_s^T \|\frac{\partial\rho}{\partial t}\|_m \exp\left(-\frac{1}{2}\int_s^T\left(\lambda^{(0)}(t) - 2\eta(t)\right)dt\right)\right)
\end{aligned}$$

**Lemma 8** Under (A.3), if $\lambda^{(0)}(t) = o(\eta(t))$, $\|\partial\rho/\partial t\|_m = o(\eta(t))$ $(t\to\infty)$ and $\int_s^\infty \eta(t)dt = \infty$, then

$$\lim_{T\to\infty}\widehat{H}^{(0)}(T) = 0$$

**Corollary 9** For any bounded function $\varphi$,

$$\left|E^{\rho}_{f\cdot\mu_s}\left(\varphi(X_t)\right) - \mu_t(\varphi)\right| \le \|\varphi\|_\infty \widehat{H}^{(0)}(t)^{1/2}$$

Similarly to Fitzsimmons (1),

$$\int |u - m_C(u)|^2 g(x)dm(x) \le k\|R_C g\|_\infty \mathcal{E}(u,u), \ u\in\mathcal{F}$$

for suitable (compact) set $C$, where $m_C = m/m(C)$ and

$$R_C g(x) = E_x\left(\int_0^\infty \exp\left(-\int_0^t I_C(X_\tau)d\tau\right)g(X_t)dt\right)$$

In particular, if $\mathbf{M}$ is ergodic , then we can take $D = X$. In this case,

$$\int |u - \mu_t(u)|^2 \rho^2(t,x) dm(x) \le \frac{\|R_1 g\|_\infty}{\inf_{x\in X} \rho^2(t,x)} \mathcal{E}^{(t)}(u,u)$$

Hence we can take $\eta$ as

$$\eta_0 = \frac{\inf_{x\in X} \rho^2(t,x)}{\|R_1 g\|_\infty}$$

**Case of Example 5**

Supose that $\mathbf{M}$ is ergodic and $R_1$ has bounded density. Then $R_1\rho^2(t,\cdot) \le \left(\sup_{x,y} r_1(x,y)\right)$. Hence, if we consider the process $\mathbf{M}^\rho_c$ corresponding to $U \wedge c$ instead of $U$, then

$$\begin{aligned} \eta_0(t) &= \frac{k}{Y(t)} \frac{1}{1+t}, \quad \lambda_0(t) = \frac{1}{1+t} - \frac{d \log Y}{dt} \\ \|\frac{\partial \rho}{\partial t}\|_m &= \frac{\mathrm{Var}_t(U)}{1+t}, \quad \mathrm{Var}_t(U) = \int (U(x) - \mu_t(U))^2 d\mu_t \end{aligned}$$

Hence , noting that $Y(t), \mathrm{Var}_t(U) \to 0$, it holds that
$\lambda_0(t) = o(\eta_0(t))$, $\|\partial\rho/\partial t\|_m = o(\eta_0(t))$ and $\int_s^\infty \eta_0(t)dt = \infty$. Hence $\widehat{H}^{(0)}(t) \to 0$ as $t \to \infty$. In the case of Example 5, we can summerize as follows: Put

$$m = \max_{x,y} \min_{0\le t\le 1} (U(\xi(t)) - U(x) - U(y))$$

($\xi(t)$:curve connecting $x, y$).

(1) There is no connected set of the form $D = \{U(x) \le b\}$ such that $D \cap \{x : U(x) = 0\} = \emptyset$, and
$b - \min\{U(x) : x \in D\} > m$.

(2) If $c > m$, $\mathbf{M}^\rho$ approximately concentrate on the set of the form $\{x : U(x) \le c\}$ which contains a point $x$ such that $U(x) = 0$.

(3) $\mathbf{M}^\rho$ concentrate at the mimimal points of $U$

**Remark** The choice of $\eta$ taken above is not optimal. In fact, Holley, Kusuoka and Stroock have shown that the spectral gap $\eta(t) \sim (1+t)^{-m/c}$.