On the reachability to some moving domains of time dependent diffusion processes

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1. Preliminaries

Let $X = R^d$, $dm(x) = dx$ and $(\mathcal{E}^t, \mathcal{F})$ be the time dependent Dirichlet forms on $L^2(X; m)$ determined by

$$\mathcal{E}^t(u, v) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

Assume that

$$\begin{aligned} \underline{a}(t) \sum_{i,j=1}^{d} a_{ij}^{(0)}(x) \xi_i \xi_j &\leq \sum_{i,j=1}^{d} a_{ij}(t, x) \xi_i \xi_j \\ &\leq \frac{1}{\underline{a}(t)} \sum_{i,j=1}^{d} a_{ij}^{(0)}(x) \xi_i \xi_j \end{aligned}$$

for a non-increasing positive function $\underline{a}(t)$ and positive definite family $(a_{ij}^{(0)}(x))$. $\mathbf{M} = \left(P_{(s,x)}, X_t\right)$ be the corresponding diffusion process and $Z_t = (\tau(t), X_t)$ with $\tau(t) = \tau(0) + t$. Fix a domain $\Gamma = \{(t, x) : |x| \geq r(t)\}$ for non-decreasing function $r(t)$ and put $\sigma_\Gamma = \inf\{t > 0 : Z_t \in \Gamma\}$.

Problem

(1) To give a condition on $r(t)$ under which $P_{(s,x)}(\sigma_\Gamma < \infty) = 1$.

(2) For a neighbourhood C of 0, to give a condition on $r(t)$ under which

$$\lim_{s \to \infty} P_{(s, \varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C) = 0.$$

Note that (2) does not hold if X_t is "transient".

2. General criterion for (1) and its application

Let $\widehat{X}_t$ be the dual process of X_t

$$
\begin{aligned}
\widehat{Z}_t &= \left(\widehat{\tau}(t), \widehat{X}_t\right), \quad \widehat{\tau}(t) = \widehat{\tau}(0) - t \\
\widehat{\sigma}_\Gamma &= \inf\{t > 0 : \widehat{Z}_t \in \Gamma\}, \quad \Gamma \subset [0, \infty) \times X
\end{aligned}
$$

Then

$$\widehat{u}_\Gamma(t, y) = \widehat{E}_{(t,y)}\left(\varphi_0(\widehat{X}_t) : t < \widehat{\sigma}_\Gamma\right)$$

satisfies

$$\int_X \frac{\partial \widehat{u}_\Gamma}{\partial t}(t, x)\varphi(x)dm(x) + \mathcal{E}^t\left(\varphi, \widehat{u}_\Gamma(t, \cdot)\right) = 0$$

Let $\Phi(t, \cdot)$ be a one to one smooth mapping from R^d onto itself such that $\Phi(0, y) = y$ and put

$$
\begin{aligned}
\gamma_{ij}(t, y) &= \frac{\partial \Phi_i}{\partial y_j} \quad \left(\gamma_{ij}^{-1}\right) = (\gamma_{ij})^{-1} \\
\alpha_{ij}(t, y) &= \sum_{i,j=1}^{d} \gamma_{ki}^{-1} a_{kl}\left(t, \Phi(t)\right) \gamma_{lj}^{-1} \\
\rho(t, y) &= \det\left(\gamma_{ij}\right) \\
\beta_i(t, y) &= \sum_{i,j=1}^{d} \gamma_{ki}^{1-} \frac{\partial \Phi_k}{\partial t} \\
\mathcal{E}^{Y,t}(\varphi, \psi) &= \sum_{i,j=1}^{d} \int_D \alpha_{ij}(t, y) \frac{\partial \varphi}{\partial y_i} \frac{\partial \psi}{\partial y_j} \rho(t, y) dy
\end{aligned}
$$

Let $Y_t = \Phi(t, X_t)$ $\widehat{Y}_t = \Phi(t, \widehat{X}_t)$. Then they are in dual in $L^2(\mu_t)$, $\mu_t(dy) = \rho(t, y)dy$. For a domain D, put

$$\widehat{\varphi}_F(t, y) = \widehat{P}_{(t,y)}\left(\varphi_0(\widehat{Y}_t) : t < \sigma_F\right), \quad F = X \setminus D.$$

Then it satisfies $\widehat{\varphi}_F(0, y) = f(y)$, $\widehat{\varphi}_F(t, y) = 0$ for $y \in F$ and

$$
\begin{aligned}
&\int_D \psi(t, y) \frac{\partial(\widehat{\varphi}_F \rho)}{\partial t}(t, y)dy + \mathcal{E}^{Y,t}(\psi, \widehat{\varphi}_F) \\
&+ \sum_{i=1}^{d} \int_D \beta_i(t, y) \frac{\partial \psi}{\partial y_i} \widehat{\varphi}_F(t, y)\rho(t, y)dy = 0.
\end{aligned}
$$

Put $\psi = \widehat{\varphi}_F$ and

$$\widehat{H}_F(t) = \int_{D_0} \widehat{\varphi}_F^2(t,y)\rho(t,y)dy.$$

Then

$$\begin{aligned}
\mathcal{E}^{Y,t}\left(\widehat{\varphi}_F, \widehat{\varphi}_F\right) &= -\frac{1}{2}\frac{d}{dt}\int_D \widehat{\varphi}^2(t,y)\rho(t,y)dy - \frac{1}{2}\int_D \frac{\partial \rho}{\partial t}\widehat{\varphi}_F^2 dy \\
&\quad -\frac{1}{2}\sum_{i=1}^d \int_D \beta_i(t,y)\frac{\partial \widehat{\varphi}_F^2}{\partial t}\rho(t,y)dy \\
&= -\frac{1}{2}\frac{d\widehat{H}_F}{dt} + \frac{1}{2}\int_D \left(\sum_{i=1}^d \frac{\partial(\beta_i \rho)}{\partial y_i} - \frac{\partial \rho}{\partial t}\right)\widehat{\varphi}_F^2(t,y)dy \\
&\le -\frac{1}{2}\frac{d\widehat{H}_F}{dt} + \frac{1}{2}\eta(t)\widehat{H}_F(t)
\end{aligned}$$

for

$$\eta(t) = \sup_{y\in D}\frac{1}{\rho(t,y)}\left(\sum_{i=1}^d \frac{\partial(\beta_i^{(1)}\rho)}{\partial y_i} - \frac{\partial \rho}{\partial t}\right).$$

Assume that there exists non-increasing function $\bar{\alpha}(t)$ and a positive definite $\left(\alpha_{ij}^{(0)}\right)$ such that

$$\underline{\alpha}(t)\sum_{i,j=1}^d \alpha_{ij}^{(0)}(y)\xi_i\xi_j \le \alpha_{ij}(t,y)\xi_i\xi_j.$$

Furthermore assume that there exists a constant $k_0 > 0$ such that

$$k_0\int_D \varphi(y)^2 dy \le \sum_{i,j=1}^d \int_D \alpha_{ij}^{(0)}(y)\frac{\partial \varphi}{\partial y_i}\frac{\partial \varphi}{\partial y_j}dy, \quad \forall \varphi \in H_0^1(D) \tag{1}$$

Then

$$\frac{d\widehat{H}_F}{dt} \le (\eta(t) - 2\lambda(t))\,\widehat{H}_F(t)$$

for

$$\lambda(t) = k_0\underline{\alpha}(t)\frac{\inf_{y\in D}\rho(t,y)}{\sup_{y\in D}\rho(t,y)}.$$

Lemma 1 Under our settings,

$$\widehat{H}_F(t) \le \|\varphi_0\|_{\mu_0}^2 \exp\left(\int_0^t (\eta(\tau) - 2\lambda(\tau))\,d\tau\right). \tag{2}$$

Note that if

$$b(t)\sum_{i,j=1}^{d}\alpha_{ij}^{(0)}(y)\xi_i\xi_j \le \sum_{i,j=1}^{d} a_{ij}^{(0)}(\Phi(t,y))\xi_i\xi_j$$

for a non-increasing function $b(t)$, then we can take

$$\underline{\alpha}(t) = \frac{1}{r(t)^2}\underline{a}(t)b(t).$$

Put $\Gamma = \{(t, \Phi(t,y)) : t \ge 0, y \in F\}$ and

$$\widehat{u}_\Gamma(t,x) = \widehat{E}^X_{(t,x)}\left(\varphi_0(\widehat{X}_t) : t < \sigma_\Gamma\right).$$

Since $P_{(0,\varphi_0\cdot m)}(T < \sigma_F) = P^Y_{(0,\varphi_0\cdot\mu_0)}(T < \sigma_\Gamma)$,

$$\begin{aligned} P^Y_{(0,\varphi_0\cdot\mu_0)}(T < \sigma_F) &= \int_D \widehat{\varphi}_F(t,y)d\mu_T(y) \le \sqrt{\mu_T(D)}\widehat{H}(T)^{1/2} \\ &\le \|\varphi_0\|_{\mu_0}\sqrt{\mu_T(D)}\exp\left(\int_0^t\left(\frac{1}{2}\eta(\tau) - \lambda(\tau)\right)d\tau\right) \\ &\to 0, \quad T \to \infty. \end{aligned}$$

Hence we have

Theorem 2 Under (1), if

$$\lim_{T\to\infty}\sqrt{\mu_T(D)}\exp\left(\int_0^T\left(\frac{1}{2}\eta(\tau) - \lambda(\tau)\right)d\tau\right) = 0,$$

then $P_{\varphi_0\cdot m}(\sigma_\Gamma < \infty) = 1$. Apply this theorem for $\Phi_i(t,y) = r(t)y_i$ and $F = \{y : |y| \ge 1\}$. Then

$$\begin{aligned} \gamma_{ij}(t,y) &= r(t)\delta_{ij} \\ \rho(t,y) &= r(t)^d \\ \alpha_{ij}(t,y) &= \frac{1}{r(t)^2}a_{ij}(t,\Phi(t,y)) \\ \lambda(t) &= k_0\frac{\underline{a}(t)}{r(t)^2} \\ \eta(t) &= \mathrm{div}\beta - d\frac{r'(t)}{r(t)} = 0. \end{aligned}$$

Hence, applying Theorem 2, we have the following theorem.

Theorem 3 Suppose that

$$\lim_{T\to\infty} r(T)^{1/2} \exp\left(-k_0 \int_0^T \frac{\underline{\alpha}(\tau)}{r(\tau)^2} d\tau\right) = 0. \tag{3}$$

Then

$$P_{(0,\varphi_0\cdot m)}(\sigma_\Gamma < \infty) = 1. \tag{4}$$

In particular, if $a_{ij} = \frac{1}{2}\delta_{ij}$, then (3) is satisfied if

$$r(t) \leq c\sqrt{t} \tag{5}$$

for large t with $c < \sqrt{d/(4k_0)}$.

In the case of Brownian motion,

$$\begin{aligned} P_0\left(|X_t| > r(t)\right) &= \frac{1}{(\sqrt{2\pi t})^d} \int_{\{|x|>r(t)\}} e^{-|x|^2/(2t)} dx \\ &= \frac{S(1)}{(\sqrt{2\pi}^d)} \int_{r(t)/\sqrt{t}}^{\infty} y^{d-1} e^{-y^2/2} dy \\ &\leq \left(k_1 + k_2 \left(\frac{r(t)}{\sqrt{t}}\right)^{d-2}\right) e^{-r(t)^2/(2t)} \end{aligned}$$

Hence, if $r(t) \geq kt^p$ for large t for some $p > \frac{1}{2}$, then

$$\int_0^\infty P_0\left(|X_t| > r(t)\right) dt < \infty.$$

2. General criterion for (2) and its application

Assume that $\mathcal{E}^{(0)}$ is recurrent. For $\Gamma \subset [0,\infty) \times X$, h: excessive function of Z_t $(h \cdot I_\Gamma \in L^2(dtdm))$

$$H_\Gamma h(z) \equiv E_z\left(h(Z_{\sigma_\Gamma})\right).$$

$H_\Gamma h =$ q.c.version of $e_{h\cdot I_\Gamma} = \lim_{\epsilon\to 0} h_\epsilon$:

$$-\int_X \frac{\partial h_\epsilon}{\partial t}(t,x)v(x)dm(x) + \mathcal{E}^t\left(h_\epsilon(t,\cdot), v\right) = \frac{1}{\epsilon}\int_X (h_\epsilon - hI_\Gamma)^-(t,x)v(x)dm(x)$$

Since $h_\epsilon \leq h$ on Γ,

$$\int_X \left(h_\epsilon - hI_\Gamma\right)^- (t,x) \left(h_\epsilon - g\right)(t,x) dm(x) \leq 0,$$

for $g \geq 0$, $g = h$ on Γ. Hence

$$\begin{aligned}
&-\int_X \frac{\partial h_\epsilon}{\partial t}(t,x)(h_\epsilon - g)(t,x)dm(x) + \mathcal{E}^t\left(h_\epsilon(t,\cdot),(h_\epsilon - g)(t,\cdot)\right) \\
&= \; -\frac{1}{2}\int_X \frac{\partial (h_\epsilon - g)^2}{\partial t}(t,x)dm(x) - \int_X \frac{\partial g}{\partial t}(t,x)(h_\epsilon - g)(t,x)dm(x) \\
&\quad +\mathcal{E}^t\left(h_\epsilon(t,\cdot),(h_\epsilon - g)(t,\cdot)\right) \\
&\leq \; 0. \qquad (6)
\end{aligned}$$

Multiply $\frac{1}{\underline{a}(t)}$ and integrate over $t \in [t_1, t_2]$:

$$\begin{aligned}
0 \;\geq\; & \frac{1}{2\underline{a}(t_1)}\|(h_\epsilon - g)(t_1,\cdot)\|_m^2 - \frac{1}{2\underline{a}(t_2)}\|(h_\epsilon - g)(t_2,\cdot)\|_m^2 \\
&+\frac{1}{2}\int_{t_1}^{t_2}\left(\int_X (h_\epsilon - g)^2(t,x)dm(x)\right)\frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right)dt \\
&-\int_{t_1}^{t_2}\frac{1}{\underline{a}(t)}\left(\int_X \frac{\partial g}{\partial t}(t,x)(h_\epsilon - g)(t,x)dm(x)\right)dt \\
&+\int_{t_1}^{t_2}\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(h_\epsilon(t,\cdot),(h_\epsilon - g)(t,\cdot)\right)dt.
\end{aligned}$$

If $g(t,x)$ is non-increasing relative to t on $[t_1,t_2]$, by letting $\epsilon \to 0$,

$$\begin{aligned}
0 \;\geq\; & -\frac{1}{2\underline{a}(t_2)}\int_{X\setminus\Gamma_{t_2}}(H_\Gamma h)^2(t_2,x)dm - \frac{1}{\underline{a}(t_1)}\int_{X\setminus\Gamma_{t_1}}(g\cdot H_\Gamma h)(t_1,x)dm \\
&-\frac{1}{2}\int_{t_1}^{t_2}\left(\int_{X\setminus\Gamma_t} g^2(t,x)dm(x)\right)\frac{d}{dt}\frac{1}{\underline{a}(t)}dt \\
&+\int_{t_1}^{t_2}\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(H_\Gamma h(t,\cdot),(H_\Gamma h - g)(t,\cdot)\right)dt
\end{aligned}$$

Lemma 4 Suppose that $g(t,x) = h(t,x)$ on Γ, $g(\cdot,x)$ and $\underline{a}$ are non-increasing on $[t_1,t_2]$. Then

$$\begin{aligned}
&-\frac{1}{2\underline{a}(t_2)}\int_{X\setminus\Gamma_{t_2}}(H_\Gamma h)^2(t_2,x)dm(x) \\
&\quad -\frac{1}{\underline{a}(t_1)}\int_{X\setminus\Gamma_{t_1}}(g\cdot H_\Gamma h)(t_1,x)dm(x) \\
&\quad +\int_{t_1}^{t_2}\mathcal{E}^t\left(H_\Gamma h(t,\cdot),H_\Gamma h(t,\cdot)\right)dt \\
&\leq \; \frac{1}{2}\int_{t_1}^{t_2}\left(\int_{X\setminus\Gamma_t} g^2(t,x)dm(x)\right)\frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right)dt \\
&\quad +\int_{t_1}^{t_2}\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(H_\Gamma h(t,\cdot),g(t,\cdot)\right)dt. \qquad (7)
\end{aligned}$$

$C = \{x : |x| \le \ell\}$ $(0 < \ell < 1)$, $B_R = \{x : |x| \le R\}$.
$\mathbf{M}^{T,C} = (X_t, P^{T,C}_{(s,x)})$: process corresponding to the Dirichlet form $\mathcal{E}^t$ with reflecting boundary $\partial B_{r(T)+1}$ and absorbing boundary ∂C.

$\xi_T(t)$: non-increasing function such that $\xi_T(t) = 1$ for $t \le T-1$
$\xi_T(t) = 0$ for $t \ge T$.

Then $h(t,x) \equiv \xi_T(t)$ is an excessive function relative to $\mathbf{M}^{T,C}$.

$g_T(t,x)$: non-increasing function such that $g_T(t,x) = \xi_T(t)$ on Λ
$g_T(t,x) = 0$ for $x \in C$, Apply Lemma 4 for $\Lambda_T = \{(t,x) : t \le T, r(t) < |x| \le r(T)+1\}$.

$$
\begin{aligned}
&-\frac{1}{2\underline{a}(t_1)}\int_{X\setminus\Lambda_{t_1}} (H_\Lambda \xi_T)^2(t_1,x)dm(x) \\
&\quad -\frac{1}{2\underline{a}(t_2)}\int_{X\setminus\Lambda_{t_2}} (H_\Lambda \xi_T)^2(t_2,x)dm(x) \\
&\quad +\int_{t_1}^{t_2}\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(H_\Lambda\xi_T(t,\cdot), H_\Lambda\xi_T(t,\cdot)\right)dt \\
\le\ & \frac{1}{2}\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right)\left(\int_{X\setminus\Lambda_t} g_T^2(t,x)dm(x)\right)dt \\
&\quad +\int_{t_1}^{t_2}\frac{1}{\underline{a}(t,x)}\mathcal{E}^t\left(H_\Lambda\xi_T(t,\cdot), g_T(t,\cdot)\right)dt.
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &\equiv \lim_{T\to\infty}\Lambda_T = \{(t,x) : r(t) < |x|\} \\
u_\Gamma(t,x) &\equiv \lim_{T\to\infty} H_\Lambda\xi_T(t,x) = P^X_{(t,x)}\left(\sigma_\Gamma < \sigma_C\right) \\
g(t,x) &\equiv \lim_{T\to\infty} g_T(t,x)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{S\to\infty}\frac{1}{S-s}\left(-\frac{1}{2\underline{a}(S)}\int_{X\setminus\Gamma_S} u_\Gamma(S,x)^2dm(x)\right. \\
&\qquad \left.+\int_s^S\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(u_\Gamma(t,\cdot), u_\Gamma(t,\cdot)\right)dt\right) \\
\le\ & \frac{1}{2}\lim_{S\to\infty}\frac{1}{S-s}\int_s^S\frac{d}{dt}\underline{a}(t)\left(\int_{X\setminus\Lambda_t} g_T^2(t,x)dm(x)\right)dt \\
&\quad +\lim_{S\to\infty}\frac{1}{S-s}\int_s^S\frac{1}{\underline{a}(t)}\mathcal{E}^t\left(u_\Gamma(t,\cdot), g(t,\cdot)\right)dt.
\end{aligned}
$$

Hence, if

$$\lim_{S\to\infty}\frac{1}{\underline{a}(S)(S-s)}\int_{X\setminus\Gamma_S}u_\Gamma(S,x)^2dm(x)=0 \tag{8}$$

$$\lim_{S\to\infty}\frac{1}{S-s}\int_s^S\frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right)\left(\int_{X\setminus\Gamma_t}g^2(t,x)dm(x)\right)dt=0 \tag{9}$$

$$\lim_{S\to\infty}\frac{1}{S-s}\int_s^S\frac{1}{\underline{a}(t)}\mathcal{E}^{(X,t)}\left(g(t,\cdot),g(t,\cdot)\right)dt=0 \tag{10}$$

then

$$\lim_{S\to\infty}\frac{1}{S-s}\int_s^S\frac{1}{\underline{a}(t)}\mathcal{E}^{t}\left(u_\Gamma(t,\cdot),u_\Gamma(t,\cdot)\right)dt=0. \tag{11}$$

Put

$$\bar{a}(t,r)=\sum_{i,j=1}^d\int_{\partial B_1}a_{ij}(t,\theta r)\theta_i\theta_j d\sigma(\theta)$$

∂B_1: unit of radius 1, $\quad d\sigma$: surface measure

$$\psi_t(r)=\begin{cases}0 & r\le\ell\\ \frac{1}{A(t)}\int_\ell^r\bar{a}(t,r)^{-1}r^{1-d}dr & \ell\le r\le r(t)\\ 1 & r(t)\le r\end{cases} \tag{12}$$

where

$$A(t)=\int_\ell^{r(t)}\bar{a}(t,r)^{-1}r^{1-d}dr.$$

Suppose

$$\bar{a}(t,r)\ \text{ and }\ A(t)\ \text{ are non-decreasing relative to } t. \tag{13}$$

Then $g(t,x)\equiv\psi_t(|x|)$ is non-increasing. Since

$$\int_{\Gamma_S}u_\Gamma(S,x)^2dm(x)\le\int_{\{x:|x|<r(S)\}}dm(x)\le k\ r(S)^d,$$

(8) is satisfied if

$$\lim_{S\to\infty}\frac{1}{S\underline{a}(S)}r(S)^d=0. \tag{14}$$

Similarly, since

$$\int_{X\setminus\Gamma_t}g^2(t,x)dx\le k\left(r(t)^d-\ell^d\right),$$

(9) holds if

$$\lim_{t\to\infty} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \times r(t)^d = 0. \tag{15}$$

since

$$\mathcal{E}^{X,t}\left(g(t,\cdot), g(t,\cdot)\right) = \frac{1}{A(t)},$$

(10) is satisfied if

$$\lim_{t\to\infty} \underline{a}(t) \times A(t) = \infty. \tag{16}$$

For the existence of $r(t)$ satisfying (16), it is necessary that $\bar{a}(t,r)$ satisfies

$$A(\infty) = \int_{\ell}^{\infty} \bar{a}(t,r)^{-1} r^{1-d} dr = \infty, \tag{17}$$

(Ichihara's test for the recurrence of $\mathcal{E}^t$).

If (14), (15) and (16) holds, then

$$\lim_{S\to\infty} \frac{1}{S-s} \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^t\left(u_\Gamma(t,\cdot), u_\Gamma(t,\cdot)\right) dt = 0.$$

Hence

$$\lim_{S\to\infty} \frac{1}{S-s} \int_s^S \int_X \sum_{i,j=1}^d a_{ij}^{(0)}(x) \frac{\partial u_\Gamma(t,x)}{\partial x_i} \frac{\partial u_\Gamma(t,x)}{\partial x_j} dx dt = 0.$$

Since $\mathcal{E}^{(0)}$ on X is recurrent, $\exists \varphi_0 \in L^1_+(m)$, $\exists K(\varphi_0)$, $\exists G \subset C$ such that

$$\int_X \left| u(x) - \frac{1}{m(G)} \langle I_G \cdot m, u\rangle \right| \varphi_0(x) dm(x) \le K(\varphi_0) \left(\mathcal{E}^{(0)}(u,u)\right)^{1/2}$$

Let $u = u_\Gamma$, then $\langle I_G \cdot m, u_\Gamma\rangle = 0$,

$$\left(\int_X u_\Gamma(t,x)\varphi_0(x) dm(x)\right)^2 \le \left(K(\varphi_0)\right)^2 \mathcal{E}^{(0)}\left(u_\Gamma(t,\cdot), u_\Gamma(t,\cdot)\right).$$

Therefore

$$\begin{aligned} & \lim_{S\to\infty} \frac{1}{S-s} \int_s^S \left(\int_{\Gamma_t} u_\Gamma(t,x)\varphi_0(x) dm(x)\right)^2 dt \\ = & \lim_{S\to\infty} \frac{1}{S-s} \int_s^S \left(P_{(t,\varphi_0\cdot m)}\left(\sigma_\Gamma < \sigma_C\right)\right)^2 dt = 0. \end{aligned}$$

If $\varphi_0 \in \mathcal{D}(\mathcal{G}^{(t)})$, for all t,

$$\begin{aligned}
&\int_X P_{(t,y)}\left(\sigma_\Gamma < \sigma_C\right)\varphi_0(y)dy - \int_X P_{(s,x)}\left(\sigma_\Gamma < \sigma_C\right)\varphi_0(x)dx \\
&= \int_s^t \int_X \mathcal{E}^\tau\left(P_{(\tau,\cdot)}\left(\sigma_\Gamma < \sigma_C\right), \varphi_0\right)d\tau \\
&\leq \int_s^t \int_X |\mathcal{G}^s\varphi_0|(x)dx
\end{aligned}$$

If $\sup_s \|\mathcal{G}^s\ \varphi_0\|_{L^1} < \infty$, then $P_{(t,\varphi_0\cdot m)}\left(\sigma_\Gamma < \sigma_C\right)$ is uniformly continuous relative to t and hence

$$\lim_{t\to\infty} P_{(t,\varphi_0\cdot m)}\left(\sigma_\Gamma < \sigma_C\right) = 0.$$

Theorem 5 Suppose that $\bar{a}(t,r)$ is non-decreasing relative to t. If there exists a function $r(t)$ satisfying (14), (15) and (16), then

$$\lim_{S\to\infty} \frac{1}{S-s}\int_s^S P_{(t,\varphi_0\cdot m)}\left(\sigma_\Gamma < \sigma_C\right)dt = 0.$$

In particular, if $\sup_s \|\mathcal{G}^s\ \varphi_0\|_{L^1} < \infty$, then

$$\lim_{s\to\infty} P_{(s,\varphi_0\cdot m)}\left(\sigma_\Gamma < \sigma_C\right) = 0.$$

If the result holds for some Γ, then it also holds for any subset of Γ.

Example 6 Suppose that $\underline{a}(t) = \underline{a}$ is independent of t. Then (15) is satisfied. Condition (14) holds if

$$r(t) = o\left(t^{1/d}\right). \tag{18}$$

Suppose that

$$\bar{a}(t,r) \leq b(t)\bar{a}_0(r)$$

for a non-decreasing function $b(t)$. Then

$$A(t) \geq \frac{1}{b(t)}\int_\ell^{r(t)} \bar{a}_0(r)^{-1}r^{1-d}dr.$$

Hence, if we can find $r(t)$ such that $r(t) = o(t^{1/d})$ and

$$\lim_{t\to\infty} \frac{1}{b(t)}\int_\ell^{r(t)} \bar{a}_0(r)^{-1}r^{1-d}dr = \infty, \tag{19}$$

then the conditions of Theorem hold. In particular, if $a_{ij}(t,x) = a_{ij}^{(0)}(x)$ with $(a_{ij}^{(0)}(x))$ satisfying (17) for

$$\bar{a}_0(r) = \sum_{i,j=1}^{d} \int_{\partial B_1} a_{ij}^{(0)}(\theta r)\theta_i\theta_j d\sigma(\theta),$$

then

$$A(t) = \int_{\ell}^{r(t)} \bar{a}_0(r)^{-1} r^{1-d} dr.$$

Hence $r(t) = t^\alpha$ with $0 < \alpha (< d/2)$ satisfies the condition of Theorem.

In the case of Brownian motion $a_{ij}(t,x) = \frac{1}{2}\delta_{ij}$. Assume $d = 1$ or 2.

$d = 1 \Longrightarrow A(t) = r(t) - \ell$
$\Longrightarrow r(t) = t^\alpha + 1$ for $0 < \alpha$ satisfies the condition of the theorem.

$d = 2 \Longrightarrow A(t) = \log r(t) - \log \ell$
$\Longrightarrow r(t) = t^\alpha$ for $0 < \alpha$ satisfies the condition of the theorem.