Boost Matrix Elements of SO(3,1) and Those of SO(4)

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(Received November 30, 2012)

The d-matrix elements of SO(4) are given by direct integration after an analytic continuation from an integral formula for the boost matrix elements of SO(3,1) and given in terms of a linear combination of $e^{-im\theta}$ with the vector addition coefficients. The boost matrix elements of SO(3,1) are given in terms of finite number of hypergeometric functions with the argument $1 - e^{-2\zeta}$ and then are expressed in terms of $e^{-2s\zeta}$. Various symmetry relations for the d- and boost matrix elements are given. The vector addition coefficient of SO(4) is given in terms of the product of the vector addition coefficient of SO(3) and the 9-j symbol.

§1. Introduction

The Lorentz group of SO(3,1) appears in the theory of special relativity. The representation theory of the SO(3,1) group is important in relativity physics and is studied in detail. In particular, the explicit expression of their boost matrix elements is sometimes needed as in the case of the d-matrix elements of SO(3) in the non-relativistic quantum mechanics. The expression of the boost matrix elements is given through an integral formula[1)] with the d-matrix elements of SO(3) and the explicit expression can be given by an integration of the integral formula. The d-matrix elements of SO(4) are obtained by an analytic continuation of the parameters in the boost matrix elements.

On the other hand, it is known that the d-matrix elements of SO(4) is given by sum of product of the vector addition coefficients of SO(3) and $e^{im\theta}$. The boost matrix elements of SO(3,1) gives the d-matrix elements through an analytic continuation of the parameters and is shown to be well-known results.

In an article, we integrate the formula continued from that of SO(3,1) and show the matrix elements of O(4), though the results are known. In appendix, the vector addition coefficient of SO(4) is given by the product of the vector addition coefficient of SO(3) and the 9-j symbol.

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§2. Integration

The integral expression for the boost matrix elements of SO(3,1) is given as follows:[1)]

$$d^{(\rho,K)}_{j'(m)j}(\zeta) = \frac{1}{2}\sqrt{(2j'+1)(2j+1)}\sqrt{\frac{\Gamma(\rho-j'+1)\Gamma(\rho+j'+2)}{\Gamma(\rho-j+1)\Gamma(\rho+j+2)}}$$
$$\times \int_0^\pi \sin\theta d^{(j')}_{K,m}(\theta)(\cosh\zeta - \cos\theta\sinh\zeta)^\rho d^{(j)}_{K,m}(\theta')d\theta, \qquad (2.1)$$

where ρ is some complex number which characterizes the representation together with K, and j, j' and m are known-numbers characterizing the representation of SO(3) and ζ denotes a real parameter of the Lorentz group and θ' is given in terms of θ

$$\cos\theta' = \frac{\cos\theta\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\theta\sinh\zeta}.$$

The d-matrix elements of SO(3) are given as follows:

$$d^{(j)}_{m',m}(\theta) = \sqrt{\frac{(j+m)!(j-m')!}{(j-m)!(j+m')!}}\frac{e^{i\pi(m'-m)}}{\Gamma(m-m'+1)}(\cos\frac{\theta}{2})^{m+m'}(\sin\frac{\theta}{2})^{m-m'}$$
$$\times F(m-j, j+m+1; m-m'+1; \sin^2\frac{\theta}{2}) \qquad (2.2)$$
$$= \sqrt{\frac{(j-m)!(j+m')!}{(j+m)!(j-m')!}}\frac{1}{\Gamma(m'-m+1)}(\cos\frac{\theta}{2})^{2j}(\tan\frac{\theta}{2})^{m'-m}$$
$$\times F(-m-j, -j+m'; m'-m+1; -\tan^2\frac{\theta}{2}). \qquad (2.3)$$

where $F(a, b; c; z)$ denotes the hypergeometric function given in App A.1. It is noted that the expressions of (2.2) and (2.3) hold for $m' \geq m$ and $m' \leq m$ due to a factor like $\Gamma(m'-m+1)$ in front of the hypergeometric function.

It follows from the expressions (2.1) and (2.3) etc. that the following relations hold:

$$d^{(\rho,K)}_{j'(m)j}(\zeta) = d^{(\rho,m)}_{j'(K)j}(\zeta),$$
$$d^{(\rho,K)}_{j'(m)j}(\zeta) = d^{(\rho,-K)}_{j'(-m)j}(\zeta),$$
$$d^{(\rho,K)}_{j'(m)j}(\zeta) = d^{(-\rho-2,K)}_{j'(m)j}(-\zeta),$$
$$d^{(\rho,K)^*}_{j'(m)j}(\zeta) = d^{(\rho^*,K)}_{j'(m)j}(\zeta). \qquad (2.4)$$

§3. d-matrix elements of SO(4)

The d-matrix elements of SO(4) are well-known in an elegant form. Let us show the matrix elements from (2.1) by an analytic continuation of the complex number

ρ to the number J of SO(4) and of the ζ to the complex number $i\theta$. Then, the expression (2.1) becomes as follows:

$$d^{(J,K)}_{j'(m)j}(\zeta) = \frac{1}{2}\sqrt{(2j'+1)(2j+1)}\sqrt{\frac{(J-j')!(J+j'+1)!}{(J-j)!(J+j+1)!}} \times \int_0^{\pi} \sin\theta d^{(j')}_{K,m}(\theta)(\cosh\zeta - \cos\theta\sinh\zeta)^J d^{(j)}_{K,m}(\theta')d\theta, \qquad (3.1)$$

where the notation ζ remains instead of the $i\theta$ in order to avoid the confusion with the integration variable θ. The ζ is written in the $i\theta$ after the integration. It is to show the following expression of the d-matrix elements of SO(4) by integration of (3.1)

$$d^{(J,K)}_{j'(m)j}(\theta) = \Sigma_{m'}(\frac{1}{2}(J+K), \frac{1}{2}(m+m'), \frac{1}{2}(J-K), \frac{1}{2}(m-m'); j', m) \times(\frac{1}{2}(J+K), \frac{1}{2}(m+m'), \frac{1}{2}(J-K), \frac{1}{2}(m-m'); j, m)e^{-im'\theta}, \quad (3.2)$$

though it is easily shown by using the relation SO(4)$\simeq$ SO(3)$\otimes$ SO(3) (App.A.5). The vector addition coefficient has various expressions and one of them is given as follows[2)]

$$(j_1, m_1, j_2, m_2; jm) = e^{i\pi(j_1-m_1)}\frac{(j+j_2-m_1)!}{(j_2-j+m_1)!} \times\sqrt{\frac{(2j+1)(j_1+j_2-j)!(j+j_2-j_1)!(j+m)!(j_1+m_1)!(j_2-m_2)!}{(j_1+j_2+j+1)!(j+j_1-j_2)!(j-j_1+j_2)!(j-m)!(j_1-m_1)!(j_2+m_2)!}} \times F(m-j, m_1-j_1, j_1+m_1+1; m_1-j-j_2, j_2-j+m_1+1; 1), \qquad (3.3)$$

The function F in (3.3) is a hypergeometric function with a unit argument defined in App.A.1.

We insert the explicit form of the d-matrix elements (2.3) into (3.1) together with the assumption $K \geq m$ which is permitted without loss of generality because of the symmetries of the d-matrix of SO(3). By using the relation

$$\cosh\zeta - \cos\theta\sinh\zeta = e^{-\zeta}\cos^2\frac{\theta}{2} + e^{\zeta}\sin^2\frac{\theta}{2},$$

and the formula of the integral

$$\int_0^{\pi/2} \cos^p\theta\sin^q\theta d\theta = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q}{2}+1)}, \qquad Re(p,q) > -1,$$

we integrate (3.1) to get

$$d^{(J,K)}_{j'(m)j}(\zeta) = \sqrt{(2j'+1)(2j+1)} \times\sqrt{\frac{(J+j'+1)!(J-j')!(j-m)!(j+K)!(j'-m)!(j'+K)!}{(J+j+1)!(J-j)!(j+m)!(j-K)!(j'+m)!(j'-K)!}}$$

$$\times\Sigma_{n_1,n_2,s}e^{i\pi(n_1+n_2)}\frac{\Gamma(-j'-m+n_1)\Gamma(-j'+K+n_1)}{\Gamma(-j'-m)\Gamma(-j'+K)\Gamma(K-m+1)n_1!}$$
$$\times\frac{\Gamma(-j-m+n_2)\Gamma(-j+K+n_2)}{\Gamma(-j-m)\Gamma(-j+K)\Gamma(K-m+1)n_2!}e^{-(J-K+m-2s-n_2)\zeta}$$
$$\times\frac{\Gamma(J-j+1)\Gamma(J+j'-K+m-n_1-n_2-s+1)\Gamma(K-m+n_1+n_2+s+1)}{\Gamma(J-j-s+1)\Gamma(J+j'+2)s!},\tag{3.4}$$

which becomes after summing over n_1, n_2 to make the hypergeometric functions

$$d\,{}^{(J,K)}_{j'(m)j}(\zeta)=\sqrt{(2j'+1)(2j+1)}$$
$$\times\sqrt{\frac{(J+j'+1)!(J-j')!(j-m)!(j+K)!(j'-m)!(j'+K)!}{(J+j+1)!(J-j)!(j+m)!(j-K)!(j'+m)!(j'-K)!}}$$
$$\times\frac{1}{\Gamma(K-m+1)\Gamma(K-m+1)}\Sigma_t$$
$$\times\frac{\Gamma(J-j+1)\Gamma(J+j'-K+m-t+1)\Gamma(K-m+t+1)}{\Gamma(J-j-t+1)\Gamma(J+j'+2)t!}$$
$$\times F(-j'-m,-j'+K,K-m+t+1;K-m+1,-J-j'+K-m+t;1)$$
$$\times F(-j-m,-j+K,-t;K-m+1,J-j-t+1;1)e^{-(J-K+m-2t)\zeta}.\tag{3.5}$$

By using the relation (A·4), we obtain

$$F\;(\;-j'-m,-j'+K,K-m+t+1;K-m+1,-J-j'+K-m+t;1)$$
$$=\frac{\Gamma(-J-j'+K-m+t)\Gamma(-J+j')}{\Gamma(-J+K+t)\Gamma(-J-m)}$$
$$\times F(-j'-m,j'-m+1,-t;K-m+1,-J-m;1),\tag{3.6}$$
$$F\;(\;-j-m,-j+K,-t;k-m+1,J-j-t+1;1)$$
$$=e^{i\pi(j+m)}\frac{\Gamma(J-j-t+1)\Gamma(-J+j)}{\Gamma(J+m-t+1)\Gamma(-J-m)}$$
$$\times F(-j-m,j-m+1,-t;K-m+1,-J-m;1).\tag{3.7}$$

(3.5) becomes by using (3.6) and (3.7) as follows:

$$d^{(J,K)}_{j'(m)j}(\zeta)=\sqrt{(2j'+1)(2j+1)}$$
$$\times\sqrt{\frac{(J+j'+1)!(J-j')!(j-m)!(j+K)!(j'-m)!(j'+K)!}{(J+j+1)!(J-j)!(j+m)!(j-K)!(j'+m)!(j'-K)!}}$$
$$\times\frac{1}{\Gamma(K-m+1)\Gamma(K-m+1)}e^{i\pi(j+K)}\frac{\Gamma(J-j+1)\Gamma(-J+j)\Gamma(-J+j')}{\Gamma(J+j'+2)\Gamma(-J-m)\Gamma(-J-m)}$$
$$\times\Sigma_t\frac{\Gamma(J+j'-K+m-t+1)\Gamma(-J-j'+K-m+t)\Gamma(K-m+t+1)}{\Gamma(J+m-t+1)\Gamma(-J+K+t)t!}$$
$$\times F(-j'-m,j'-m+1,-t;K-m+1,-J-m;1)F(j'\to j)e^{-(J-K+m-2t)\zeta}.\tag{3.8}$$

Comparing the F in (3.3) and that in (3.8), we get the following relation of the vector addition coefficient with the F in (3.8)

$$\begin{aligned} F\ (&-j-m, j-m+1, -t; K-m+1, -J-m; 1) \\ &= e^{i\pi t}\sqrt{\frac{(J+j+1)!(J-j)!(J+m-t)!(j+m)!(j-K)!t!}{(2j+1)(J-K-t)!(K-m+t)!(j-m)!(j+K)!}} \\ &\quad\times\frac{(K-m)!}{(J+m)!}(\frac{1}{2}(J+K), \frac{1}{2}(J-K)+m-t, \frac{1}{2}(J-K), -\frac{1}{2}(J-K)+t; j, m). \end{aligned} \tag{3.9}$$

We obtain the desired relation (3.2) by inserting (3.9) into (3.8) together with the replacements $J-K+m-2t=m'$, $\zeta \to i\theta$, and using the following relation which is easily seen from the relation $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$,

$$\begin{aligned} &e^{i\pi(j+m)}\frac{\Gamma(-J-j'+K-m+t)\Gamma(J+j'-K+m-t+1)}{\Gamma(-J+K+t)\Gamma(J-K-t+1)} \\ &\quad\times\frac{\Gamma(J-j'+1)\Gamma(-J+j')}{\Gamma(J+m+1)\Gamma(-J-m)}\frac{\Gamma(J-j+1)\Gamma(-J+j)}{\Gamma(J+m+1)\Gamma(-J-m)}=1. \end{aligned}$$

The relation corresponding to the symmetry of the first of (2.4) holds due to the following relation from (3.6)

$$\begin{aligned} F\ (&-j-m, j-m+1, -t; K-m+1, -\rho-m; 1) \\ &= \frac{\Gamma(-\rho-m)\Gamma(t+1)}{\Gamma(-\rho-K)\Gamma(K-m+t+1)}\frac{\Gamma(K-m+1)\Gamma(j-K+1)\Gamma(j+m+1)}{\Gamma(m-K+1)\Gamma(j-m+1)\Gamma(j+K+1)} \\ &\quad\times F(-j-K, j-K+1, m-K-t; m-K+1, -\rho-K; 1). \end{aligned}$$

It is evident that the representation condition is satisfied as seen from (3.2):

$$\Sigma_{j''} d^{(J,K)}_{j'(m)j''}(\theta_1) d^{(J,K)}_{j''(m)j}(\theta_2) = d^{(J,K)}_{j'(m)j}(\theta_1+\theta_2).$$

Though the orthogonal relation of (3.2) is known in the form,

$$\Sigma_m \int_0^\pi \sin^2\theta \overline{d^{(J',K')}_{j'(m)j}(\theta)} d^{(J,K)}_{j'(m)j}(\theta) = \frac{\pi}{2}\frac{(2j'+1)(2j+1)}{(J+K+1)(J-K+1)}\delta_{J',J}\delta_{K',K}, \tag{3.10}$$

we show it explicitly by direct integration. The symmetry properties of vector addition coefficients and some properties of the Racah coefficients are given in (A·5)∼(A·10)[2] :

We rewrite the d-matrix elements (3.2) by using (A·10) as follows:

$$\begin{aligned} d^{(J,K)}_{j'(m)j}(\theta) &= \Sigma_s e^{i\pi(J_+-M_++2J_--s)}\sqrt{(2j'+1)(2j+1)} \\ &\times(j'm, j-m; s0)(J_-M_-, J_--M_-; s0)W(j'jJ_-J_-; sJ_+)e^{-im\theta+2iM_-\theta}, \end{aligned} \tag{3.11}$$

where $J_\pm = (J \pm K)/2, M_\pm = (m \pm m')/2$. The following expression is to be integrated:

$$T = \Sigma_m \int_0^\pi \sin^2\theta \overline{d^{(J',K')}_{j'(m)j}(\theta)} d^{(J,K)}_{j'(m)j}(\theta) d\theta$$
$$= \Sigma_{m,M_-,M'_-,M_+,M'_+} \Sigma_{s',s} e^{i\pi(J_+ - J'_+ - M_+ + M'_+ + 2J_- - 2J'_- + s' - s)} (2j'+1)(2j+1)$$
$$\times (j'm, j-m; s0)(j'm, j-m; s'0)(J_-M_-, J_- - M_-; s0)(J'_-M'_-, J'_- - M'_-; s'0)$$
$$\times W(j'jJ_-J_-; sJ_+) W(j'jJ'_-J'_-; s'J'_+) \int_0^\pi \sin^2\theta e^{2i(M_- - M'_-)\theta} d\theta,$$

which becomes

$$T = \frac{\pi}{2} \Sigma_{M_-,M'_-,s} e^{i\pi(J_+ - J'_+ + M_- - M'_- + 2J_- - 2J'_-)} (2j'+1)(2j+1)$$
$$\times (J_-M_-, J_- - M_-; s0)(J'_-M'_-, J'_- - M'_-; s0)$$
$$\times W(j'jJ_-J_-; sJ_+) W(j'jJ'_-J'_-; sJ'_+)$$
$$\times e^{i(M_- - M'_-)\pi} \frac{1}{\Gamma(2 + M_- - M'_-)\Gamma(2 - M_- + M'_-)},$$

where the following formula is used :

$$\int_0^\pi \sin^\alpha\theta e^{i\beta\theta} d\theta = \frac{\pi\Gamma(1+\alpha)e^{i\pi\beta/2}}{2^\alpha \Gamma(1 + (\alpha+\beta)/2)\Gamma(1 + (\alpha-\beta)/2)}, \qquad Re(\alpha) > -1.$$

Now, we obtain the following from (A·10)

$$\Sigma_{M_-,M'_-} (J'_-M'_-, J'_- - M'_-; s0)(J_-M_-, J_- - M_-; s0) \frac{1}{\Gamma(2+M)\Gamma(2-M)}$$
$$= \Sigma_{M_-,M'_-,M,t} e^{i\pi(2J'_- + M + s + t)} (2s+1)$$
$$\times (J'_- - M'_-, J_-M_-; tM)(J'_- - M'_-, J_-M_-; tM)$$
$$\times W(J'_-J'_-J_-J_-; st) \frac{1}{\Gamma(2+M)\Gamma(2-M)},$$

which becomes

$$\Sigma_{M_-,M'_-} (J'_-M'_-, J'_- - M'_-; s0)(J_-M_-, J_- - M_-; s0) \frac{1}{\Gamma(2+M)\Gamma(2-M)}$$
$$= \Sigma_{M,t} e^{i\pi(2J'_- + M + s + t)} (2s+1)$$
$$\times W(J'_-J'_-J_-J_-; st) \frac{1}{\Gamma(2+M)\Gamma(2-M)}.$$

However, it is easily seen that the following holds for the possible values of t,

$$\Sigma_M e^{i\pi M} \frac{1}{\Gamma(2+M)\Gamma(2-M)} = \delta_{t,0}.$$

Thus, we get

$$T = \frac{\pi}{2} \Sigma_s e^{i\pi(J_+ - J'_+ + 2J_- - 2J'_-)} (2j'+1)(2j+1)$$
$$\times W(j'jJ_-J_-; sJ_+) W(j'jJ'_-J'_-; sJ'_+) \frac{2s+1}{2J_- + 1} \delta_{J_-,J'_-}.$$

By using the relation in (A.3.4), we get our result:

$$T = \frac{\pi}{2}\frac{(2j'+1)(2j+1)}{(2J_+ + 1)(2J_- + 1)}\delta_{J_+,J'_+}\delta_{J_-,J'_-}$$
$$= \frac{\pi}{2}\frac{(2j'+1)(2j+1)}{(J+K+1)(J-K+1)}\delta_{J,J'}\delta_{K,K'}.$$

§4. Integration of (2.1)

We integrate (2.1) directly and give a useful expression. By using the following

$$(\cosh\zeta - \cos\theta\sinh\zeta)^{\rho-j} = e^{(\rho-j)\zeta}\Sigma_s\frac{\Gamma(j-\rho+s)}{\Gamma(j-\rho)s!}(1-e^{-2\zeta})^s(\cos\frac{\theta}{2})^{2s},$$

and inserting (2.3) into (2.1), we integrate the result to get

$$\begin{aligned}
{}^b d^{(\rho,K)}_{j'(m)j}(\zeta) &= \sqrt{(2j'+1)(2j+1)}\sqrt{\frac{\Gamma(\rho-j'+1)\Gamma(\rho+j'+2)}{\Gamma(\rho-j+1)\Gamma(\rho+j+2)}}\\
&\times\sqrt{\frac{(j'-m)!(j'+K)!(j-m)!(j+K)!}{(j'+m)!(j'-K)!(j+m)!(j-K)!}}\Sigma_{n_1,n_2}e^{i\pi(n_1+n_2)}\\
&\times\frac{\Gamma(-j'-m+n_1)\Gamma(-j'+K+n_1)}{\Gamma(-j'-m)\Gamma(-j'+K)\Gamma(K-m+n_1+1)n_1!}\\
&\times\frac{\Gamma(-j-m+n_2)\Gamma(-j+K+n_2)}{\Gamma(-j-m)\Gamma(-j+K)\Gamma(K-m+n_2+1)n_2!}\\
&\times\frac{\Gamma(K-m+n_1+n_2+1)\Gamma(j+j'-K+m-n_1-n_2+1)}{\Gamma(j+j'+2)}\\
&\times F(j-\rho,j+j'-K+m-n_1-n_2+1;j+j'+2;1-e^{-2\zeta})e^{(\rho-2j+K-m+2n_2)\zeta},
\end{aligned} \tag{4.1}$$

where the sum over s is written by the hypergeometric function. It follows from (4.1) that the boost matrix elements are expressed in terms of finite number of hypergeometric functions.

Rewriting (4.1) by using the relation (A.1.3) in order to see the relation between (3.2) and (4.1), we obtain the following expression

$$\begin{aligned}
{}^b d^{(\rho,K)}_{j'(m)j}(\zeta) &= \sqrt{(2j'+1)(2j+1)}\sqrt{\frac{\Gamma(\rho-j'+1)\Gamma(\rho+j'+2)}{\Gamma(\rho-j+1)\Gamma(\rho+j+2)}}\\
&\times\sqrt{\frac{(j'-m)!(j'+K)!(j-m)!(j+K)!}{(j'+m)!(j'-K)!(j+m)!(j-K)!}}\frac{1}{\Gamma(K-m+1)\Gamma(K-m+1)}\\
&\times\Sigma_{n_1,n_2}e^{i\pi(n_1+n_2)}\frac{\Gamma(-j'-m+n_1)\Gamma(-j'+K+n_1)\Gamma(K-m+1)}{\Gamma(-j'-m)\Gamma(-j'+K)\Gamma(K-m+n_1+1)n_1!}\\
&\times\frac{\Gamma(-j-m+n_2)\Gamma(-j+K+n_2)\Gamma(K-m+1)}{\Gamma(-j-m)\Gamma(-j+K)\Gamma(K-m+n_2+1)n_2!}
\end{aligned}$$

$$\times \Gamma(K-m+n_1+n_2+1)\Gamma(j+j'-K+m-n_1-n_2+1)$$
$$\times \Big[\frac{\Gamma(\rho-j+K-m+n_1+n_2+1)}{\Gamma(\rho+j'+2)\Gamma(K-m+n_1+n_2+1)}$$
$$\times F(j-\rho, j+j'-K+m-n_1-n_2+1; j-\rho-K+m-n_1-n_2; e^{-2\zeta})$$
$$+\frac{\Gamma(j-\rho-K+m-n_1-n_2-1)}{\Gamma(j-\rho)\Gamma(j+j'-K+m-n_1-n_2+1)}e^{-2(\rho-j+K-m+n_1+n_2+1)\zeta}$$
$$\times F\,(\rho+j'+2, K-m+n_1+n_2+1; \rho-j+K-m+n_1+n_2+2; e^{-2\zeta})\Big]$$
$$\times e^{(\rho-2j+K-m+2n_2)\zeta}. \tag{4.2}$$

The summation terms on the right side of (4.2) can be expressed in terms of the hypergeometric functions of ${}_3F_2$ and the hypergeometric functions can be rewritten by using (A.1.4). Then, the boost matrix elements become as follows:

$${}^b d^{(\rho,K)}_{j'(m)j}(\zeta) = \sqrt{(2j'+1)(2j+1)}\sqrt{\frac{\Gamma(\rho-j'+1)\Gamma(\rho-j+1)}{\Gamma(\rho+j'+2)\Gamma(\rho+j+2)}}$$
$$\times\sqrt{\frac{(j'-m)!(j'+K)!(j-m)!(j+K)!}{(j'+m)!(j'-K)!(j+m)!(j-K)!}}\frac{1}{\Gamma(K-m+1)\Gamma(K-m+1)}$$
$$\times e^{i\pi(j+j'+m+K)}\frac{\Gamma(-\rho+j')\Gamma(-\rho+j)}{\Gamma(-\rho-m)\Gamma(-\rho-m)}\Big[\Sigma_t\frac{\Gamma(-\rho-m+t)\Gamma(-K-m+t+1)}{\Gamma(-\rho-K+t)t!}$$
$$\times F(-j'-m, j'-m+1, -\rho-m+t; K-m+1, -\rho-m; 1)$$
$$\times F(j'\to j)e^{(\rho+K+m-2t)\zeta)}$$
$$-\Sigma_t\frac{\Gamma(\rho-m+t+2)\Gamma(K-m+t+1)}{\Gamma(\rho+K+t+2)t!}$$
$$\times F(-j'-m, j'-m+1, K-m+t+1; K-m+1, -\rho-m; 1)$$
$$\times F(j'\to j)e^{-(\rho+K-m+2+2t)\zeta}\Big]. \tag{4.3}$$

It follows from (4.3) that the symmetry with respect to $j \leftrightarrow j'$ holds.

When the analytic continuation of the number ρ to the integer J together with the $\zeta \to i\theta$ is made, the factor $\Gamma(-J-m+t)/\Gamma(-J-K+t)$ in the first term gives rise to the two sum terms from $t = 0$ to $J+m$ and from $J+K+1$ to ∞. It is easy to see that the latter term cancels out the second term of (4.3) and the remaining finite term becomes the d-matrix elements (3.2) of SO(4).

It follows that the exchange symmetry of K and m given in the first of (2.4) holds due to the following relations seen from (A·4)

$$F\;(\;-j-m, j-m+1, -\rho-m+t; K-m+1, -\rho-m; 1) = e^{i\pi(K-m)}$$
$$\times\frac{\Gamma(-\rho-m)\Gamma(t+1)\Gamma(K-m+1)\Gamma(j-K+1)\Gamma(j+m+1)}{\Gamma(-\rho-K)\Gamma(K-m+t+1)\Gamma(m-K+1)\Gamma(j+K+1)\Gamma(j-m+1)}$$
$$\times F(-j-K, j-K+1, -\rho-K+t; m-K+1, -\rho-K; 1), \tag{4.4}$$
$$F\;(\;-j-m, j-m+1, K-m+t+1; K-m+1, -\rho-m; 1) = e^{i\pi(K-m)}$$

$$\times \frac{\Gamma(-\rho-m)\Gamma(K-m+1)\Gamma(j-K+1)\Gamma(j+m+1)\Gamma(\rho+m-t+1)}{\Gamma(-\rho-K)\Gamma(m-K+1)\Gamma(j-m+1)\Gamma(j+K+1)\Gamma(\rho+K-t+1)}$$
$$\times F(-j-K, j-K+1, m-K+t+1; m-K+1, -\rho-K). \qquad (4.5)$$

Appendix

The hypergeometric functions, the vector addition coefficients, Racah coefficients and the vector addition coefficients of SO(4) are given in the appendix.

A.1. Hypergeometric Functions

$$F(a_1, a_2, ...; b_1, b_2, ...; z) \equiv \Sigma_n \frac{\Gamma(a_1+n)\Gamma(a_2+n)...}{\Gamma(a_1)\Gamma(a_2)...} \frac{\Gamma(b_1)\Gamma(b_2)...}{\Gamma(b_1+n)\Gamma(b_2+n)...} \frac{z^n}{n!}, \qquad (\text{A}\cdot 1)$$

$$\begin{aligned} F(a, b; c; z) &= (1-z)^{c-a-b} F(c-a, c-b; c; z) \\ &= (1-z)^{-a} F(a, c-b; c; \frac{z}{z-1}), \end{aligned} \qquad (\text{A}\cdot 2)$$

$$F\,(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z)$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-z), \qquad (\text{A}\cdot 3)$$

$$F\,(a, b, -n; e, f; 1) = e^{in\pi} \frac{\Gamma(f)\Gamma(1+b-f)}{\Gamma(1+b-n-f)\Gamma(f+n)}$$
$$\times F(e-a, b, -n; e, 1+b-n-f; 1), \quad n = 0, 1, 2, \cdots \qquad (\text{A}\cdot 4)$$

A.2. Vector addition coefficient of SO(3)

$$\begin{aligned} (j_1 m_1 . j_2 m_2; j_3 m_3) &= (-1)^{j_1+j_2-j_3} (j_2 m_2 . j_1 m_1; j_3 m_3) \\ &= (-1)^{j_1+j_2-j_3} (j_1 - m_1 . j_2 - m_2; j_3 - m_3) \\ &= (-1)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}} (j_1 m_1 . j_3 - m_3; j_2 - m_2). \end{aligned} \qquad (\text{A}\cdot 5)$$

A.3. Racah coefficient

The Racah coefficients vanish unless the triads abe,cde,bdf, and acf form a triangle.

$$W(abcd; ef) = W(badc; ef) = W(cdab; ef) = W(acbd; fe), \qquad (\text{A}\cdot 6)$$
$$= (-1)^{e+f-a-d} W(ebcf; ad) = (-1)^{e+f-b-c} W(aefd; bc). \qquad (\text{A}\cdot 7)$$

$$\Sigma(2e+1)W(abcd;ef)W(abcd;ef')=\frac{1}{2f+1}\delta_{f,f'}, \tag{A·8}$$

$$W(abcd,0f)=e^{i\pi(b+c-f)}\frac{1}{\sqrt{(2b+1)(2c+1)}}\delta_{ab}\delta_{cd}.$$

$$\begin{aligned}&W(a\alpha b\beta;c\gamma)W(a'\alpha b'\beta;c'\gamma)\\&=\Sigma(2s+1)W(a's\alpha c;ac')W(bs\beta c\prime;b'c)W(a's\gamma b;ab'),\end{aligned} \tag{A·9}$$

The Racah coefficient and the vector coefficients are related by

$$\begin{aligned}&(\ j_1m_1,j_2m_2;jm_1+m_2)(jm_1+m_2,j_3m_3;j_4m_4)\\&=\Sigma\sqrt{(2s+1)(2j+1)}(j_2m_2,j_3m_3;sm_2+m_3)(j_1m_1,sm_2+m_3;j_4m_4)\\&\times W(j_1j_2j_4j_3;js).\end{aligned} \tag{A·10}$$

The $9j$-symbol is defined by the Racah coefficients as follows

$$\begin{aligned}&\Sigma_s(-1)^{j_{21}+j_{22}-j_{23}+j_{31}+j_{32}-j_{33}}(2s+1)\\&\times W(j_{11}j_{12}j_{33}j_{23};j_{13}s)W(j_{21}j_{22}sj_{12};j_{23}j_{32})W(j_{31}j_{32}j_{11}s;j_{33}j_{21})\\&=\left\{\begin{matrix}j_{11}&j_{12}&j_{13}\\j_{21}&j_{22}&j_{23}\\j_{31}&j_{32}&j_{33}\end{matrix}\right\},\end{aligned} \tag{A·11}$$

A.4. The Properties of d-matrix elements of SO(3)

The d-matrix elements of SO(3) have the following symmetries:

$$\begin{aligned}d^{(j)}_{m',m}(\theta)&=e^{i\pi(m-m')}d^{(j)}_{-m',-m}(\theta),\\d^{(j)}_{m',m}(\theta)&=e^{i\pi(m-m')}d^{(j)}_{m,m'}(\theta),\\d^{(j)}_{m',m}(\theta)&=d^{(j)}_{-m',-m}(-\theta),\\d^{(j)*}_{m',m}(\theta)&=e^{i\pi(m-m')}d^{(j)}_{m',m}(\theta).\end{aligned}$$

A.5. Representation of SO(4)

The representation generators $D_{ij}=-D_{ji}$ of SO(4) are defined by the following commutation relations:

$$[D_{ij},D_{kl}]=i(\delta_{ik}D_{jl}+\delta_{jl}D_{ik}-\delta_{il}D_{jk}-\delta_{jk}D_{il}). \tag{A·12}$$

It is known that the group SO(4) is isomorphic to SO(3)⊗ SO(3) which means the following. If we define the quantities

$$J^{(\pm)}_i=\frac{1}{2}(\frac{1}{2}\epsilon_{ijk}D_{jk}\pm D_{i4}), \tag{A·13}$$

each of the quantities $J_i^{(\pm)}$ satisfies the commutation relations of SO(3). Thus, the representation of SO(4) can be determined by those of SO(3) which are well known:

$$F^{(\pm)}|j^{(\pm)}m^{(\pm)}> = j^{(\pm)}(j^{(\pm)}+1)|j^{(\pm)}m^{(\pm)}>$$
$$J_3^{(\pm)}|j^{(\pm)}m^{(\pm)}> = m^{(\pm)}|j^{(\pm)}m^{(\pm)}>, \qquad (A\cdot 14)$$

where $F^{(\pm)}$ denotes the invariant operator of SO(3) given as

$$F^{(\pm)} = \Sigma(J_i^{(\pm)})^2. \qquad (A\cdot 15)$$

The invariant operators of SO(4) are given by

$$F = \frac{1}{2}\Sigma D_{ij}^2, \qquad G = \frac{1}{2}D_{i4}\epsilon_{ijk}D_{jk}, \qquad (A\cdot 16)$$

they are expressed in terms of those of SO(3) as follows:

$$F = 2(F^{(+)} + F^{(-)}), \qquad G = F^{(+)} - F^{(-)}. \qquad (A\cdot 17)$$

On the other hand, the representation of SO(4) may be characterized by the following:

$$F|J_+J_-jm>= 2\{J_+(J_+ +1) + J_-(J_- +1)\}|J_+J_-jm>,$$
$$G|J_+J_-jm>= \{J_+(J_+ +1) - J_-(J_- +1)\}|J_+J_-jm>,$$
$$F^{(3)}|J_+J_-jm>= j(j+1)|J_+J_-jm>,$$
$$D_{12}|J_+J_-jm>= m|J_+J_-jm> . \qquad (A\cdot 18)$$

where

$$F^{(3)} = (D_{23})^2 + (D_{31})^2 + (D_{12})^2. \qquad (A\cdot 19)$$

It follows that the representation is defined by the group chain $SO(4) \supset SO(3)$. It is easy to show that the ket $|J_+J_-jm>=|JKjm>$ is connected with the direct product of those of SO(3):

$$|J_+J_-jm>= \Sigma(J_+m_+, J_-m_-; jm)|J_+m_+>|J_-m_-> . \qquad (A\cdot 20)$$

It follows from (A·17) that (A·20) is satisfied by the relations $J_+ = (J+K)/2, J_- = (J-K)/2$. It is noticed that the three quantities J_+J_-j are subject to the triangle rule.

The matrix elements of $D_{ij}(i, j \leq 3)$ with respect to the bases (A·18) are given in the same form as for SO(3), and it is better to give the matrix elements of D_{34} because those of $D_{i4}, i = 1, 2$ are given by D_{34} through the relation $[D_{34}, D_{23} \pm iD_{31}] = \pm(D_{14} \pm iD_{24})$. It follows from (A·20) and $D_{34} = J_3^{(+)} - J_3^{(-)}$ that the d-matrix elements (3.2) hold because of the elements of $\exp(-i\theta D_{34})$.

Of course, the bases (A·20) are orthonormalized as easily seen,

$$< J'_+J'_-j'm'|J_+J_-jm>= \delta_{J'_+J_+}\delta_{J'_-J_-}\delta_{j'j}\delta_{m'm},$$

and then the product ket can be expressed in terms of the ket $|J_+J_-jm>$

$$|J_+m_+>|J_-m_->=\Sigma_{jm}(J_+m_+,J_-m_-;jm)|J_+J_-jm>.$$

The matrix elements of $D_{23}\pm iD_{31}$ with respect to the bases (A·20) are the same as in the SO(3) and those of D_{34} are given as follows:

$$\begin{aligned}D_{34}|J_+J_-jm>=&\{2(-1)^{J_++j-J_-}\sqrt{J_+(J_++1)(2J_++1)(2j+1)}\\ &\times\Sigma_{j'}(j-m,10;j'-m)W(J_+jJ_+j';J_-1)-m\delta_{j'j}\}|J_+J_-j'm>,\end{aligned}\qquad\text{(A·21)}$$

where the quantity $W(abcd;ef)$ denotes the Racah W coefficient.

A.6. Direct product

We consider the case given by a direct sum of generators $D_{ij}=D_{ij}^{(1)}+D_{ij}^{(2)}$ which leads to the direct product representation. The bases $|J_+J_-jm>$, $|J_{1+}J_{1-}j_1m_1>$ and $|J_{2+}J_{2-}j_2m_2>$ corresponding to (A·20) are used in what follows.

The triangle rules for J_+J_-j, $J_{1+}J_{1-}j_1$ and $J_{2+}J_{2-}j_2$ are satisfied as in the (A·20) and thus the base $|J_+J_-jm>$ spans $(2J_++1)(2J_-+1)$ dimensional space. The total number of the direct product bases becomes $(2J_{1+}+1)(2J_{1-}+1)(2J_{2+}+1)(2J_{2-}+1)$. It follows from the triangle rules j_1j_2j and J_+J_-j that the quantities $J_{1+}J_{2+}J_+$and $J_{1-}J_{2-}J_-$ will be subject to the triangle rule. Then, the total number of the base $|J_+J_-jm>$ for the possible values of $J_+.J_-$ becomes as follows:

$$\begin{aligned}&\Sigma_{J_+=|J_{1+}-J_{2+}|}^{J_{1+}+J_{2+}}\ \Sigma_{J_-=|J_{1-}-J_{2-}|}^{J_{1-}+J_{2-}}(2J_++1)(2J_-+1)\\ &=(2J_{1+}+1)(2J_{2+}+1)(2J_{1-}+1)(2J_{2-}+1),\end{aligned}\qquad\text{(A·22)}$$

which agrees with those of the product bases.

Thus, the bases $|J_+J_-jm>$ of the representation of D_{ij} are related with the bases of the product representation as follows:

$$\begin{aligned}|J_+J_-jm>=&\Sigma\,(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)(j_1m_1,j_2m_2:jm)\\ &\times|J_{1+}J_{1-}j_1m_1>|J_{2+}J_{2-}j_2m_2>,\end{aligned}\qquad\text{(A·23)}$$

where the sum over j_1,j_2,m_1,m_2 is taken and the kets on the right are meant as in (A·20). The factor on the right side in (A·23) is the vector addition coefficient of SO(4) and must satisfy the following orthonormalization:

$$\begin{aligned}&\Sigma\,(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J'_+J'_-j')(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)\\ &\times(h_1m_1,j_2m_2;j'm')(j_1m_1,j_2m_2;jm)\\ &=\delta_{J'_+J_+}\delta_{J'_-J_-}\delta_{j'j}\delta_{m'm}.\end{aligned}\qquad\text{(A·24)}$$

It is noted that the conditions for $j'j$ and $m'm$ appear from the vector addition coefficient of SO(3).It is easily seen that the bases (A·23) are given by the relations corresponding to (A·20) and satisfy the relations corresponding to (A·18).

It is to show that the vector addition coefficient except for the vector addition coefficient of SO(3) is given by

$$(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)=(-1)^{2(J_{1+}-J_-)}$$

$$\times\sqrt{(2J_+ +1)(2J_- +1)(2j_1+1)(2j_2+1)}\left\{\begin{array}{ccc} j & J_- & J_+ \\ j_2 & J_{2-} & J_{2+} \\ j_1 & J_{1-} & j_{1+}\end{array}\right\}, \qquad (\mathrm{A{\cdot}25})$$

where the quantity enclosed by the square brakects is $9j$ symbol which is given in (A·11) and has the symmetry of interchange of any two rows or two columns. The orthogonality (A·24) of the vector addition coefficient is easily confirmed by direct calculation.

It is obvious that the relation $D_{ij}=D^{(1)}_{ij}+D^{(2)}_{(ij)}$ for $i,j\le 3$ leads to the same matrix elements because the vector addition coefficient of SO(3) is contained in (B.2.2) and the same fact for $D_{i4}=D^{(1)}_{i4}+D^{(2)}_{i4}$ must hold. However, the relation $[D_{34},D_{23}\pm D_{31}]=\pm(D_{14}\pm iD_{24})$ hold. Thus,it is sufficient to show that the matrix elements of $D^{(1)}_{34}+D^{(2)}_{34}$ lead to those of D_{34}. In the following, we show this fact by using the vector addition coefficient (A·25).

The matrix elements of D_{34} are given as follows:

$$< J_+J_-j'm|D_{34}|J_+J_-jm >= (-1)^{J_+-J_-+j}2\sqrt{J_+(J_++1)(2J_++1)(2j+1)}$$
$$\times\, W(J_+jJ_+j':J_-1)(j-m,10;j'-m)-m\delta_{j'j}, \qquad (\mathrm{A{\cdot}26})$$

while the matrix elements of $D^{(1)}_{34}+D^{(2)}_{34}$ are given in terms of the product bases as follows:

$$< J\,_+J_-j'm|(D^{(1)}_{34}+D^{(2)}_{34})|J_+J_-jm >$$
$$=\Sigma\,(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)(J_{1+}J_{1-}j'_1,J_{2+}J_{2-}j'_2;J_+J_-j')$$
$$\times\,(j_1m_1,j_2m_2;jm)(j'_1m_1,j'_2m_2;j'm)$$
$$\times\,[(-1)^{J_{1+}-J_{1-}+j_1}2\sqrt{J_{1+}(J_{1+}+1)(2J_{1+}+1)(2j_1+1)}W(J_{1+}j_1J_{1+}j'_1;J_{1-}1)$$
$$\times\,(j_1-m_1,10;j'_1-m_1)-m_1\delta_{j'_1j_1}]$$
$$+\,\Sigma(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)(J_{1+}J_{1-}j'_1,J_{2+}J_{2-}j'_2;J_+J_-j')$$
$$\times\,(j_1m_1,j_2m_2;jm)(j'_1m_1,j'_2m_2;j'm)$$
$$\times\,[(-1)^{J_{2+}-J_{2-}+j_2}2\sqrt{J_{2+}(J_{2+}+1)(2J_{2+}+1)(2j_2+1)}W(J_{2+}j_2J_{2+}j'_2;J_{2-}1)$$
$$\times\,(j_2-m_2,10;j'_2-m_2)-m_2\delta_{j'_2j_2}]. \qquad (\mathrm{A{\cdot}27})$$

We show the validity of (A·25) by the fact that two terms of the right side on (A·27) together are equal to the left term by rewriting the terms. It is easily seen that the m_1,m_2 terms on the right side together become the m term on the left because of the normalization (A·24) of the vector addition coefficient of SO(4). The first term of the right side on (B.2.6) can be rewritten by using the formulas (A·9):

$$\Sigma\,(-1)^{J_{1+}-J_{1-}-j_1+j'_1+j_2-j-1}\sqrt{J_{1+}(J_{1+}+1)(2J_{1+}+1)(2j_1+1)((2j+1)(2j'_1+1)}$$
$$\times\,W(J_{1+}j_1J_{1+}j'_1;J_{1-}1)W(j_1jj'_1j';j_21)(J_{1+}J_{1-}j_1,J_{2+}J_{2-}j_2;J_+J_-j)$$
$$\times\,(J_{1+}J_{1-}j'_1,J_{2+}J_{2-}j'_2;J_+J_-j')(j-m,10;j'-m). \qquad (\mathrm{A{\cdot}28})$$

We substitute the explicit form of the vector addition coefficient of SO(4) and first perform the sum over j'_1 with use of (A·9) to get

$$\Sigma\,(-1)^{2j'_1}(2j'_1+1)W(j'_1J_{1-}j's;J_{1+}j_2)W(J_{1+}j_1J_{1+}j'_1;J_{1-}1)W(j_1jj'_1j';j_21)$$

$$
\begin{aligned}
&= (-1)^{J_{1-}-J_{1+}+j_2-j'} \Sigma(2j_1'+1) \\
&\quad \times W(J_{1-}j_1'sj';J_{1+}j_2)W(1j_1'jj_2;j_1j')W(J_{1-}j_1'J_{1+}1;J_{1+}j_1) \\
&= (-1)^{J_{1-}-J_{1+}+j_2-j'} W(J_{1+}s1j;j'J_{1+})W(J_{1-}sj_1j;j_2J_{1+}).
\end{aligned}
$$

Next, we sum over j_1 and use (A·8) to give

$$
\begin{aligned}
&\Sigma\,(2j_1+1)W(j_1J_{1-}js_1;J_{1+}j_2)W(J_{1-}sj_1j;j_2J_{1+}) \\
&= (-1)^{s-s_1}\Sigma(2j_1+1)W(J_{1+}J_{1-}jj_2;j_1s_1)W(J_{1+}J_{1-}jj_2;j_1s) = \frac{1}{2s+1}\delta_{s,s_1}.
\end{aligned}
$$

and sum over j_2 with use of (A·8) to give

$$
\Sigma(2j_2+1)W(J_{2+}j_2J_-s;J_{2+}J_{1-})W(J_{2+}j_2J_-s;J_{2+}J_{1-}) = \frac{1}{2J_-+1}.
$$

The remaining term becomes

$$
\begin{aligned}
&(-1)^{j-j'-1}\sqrt{J_{1+}(J_{1+}+1)(2J_{1+}+1)(2j+1)}(2J_++1) \\
&\times(-1)^{2s}(2s+1)W(J_{1+}s1j;j'J_{1+})W(j'J_-J_{1+}J_{2+};J_+s) \\
&\times W(jJ_-J_{1+}J_{2+};J_+s)(j-m,10;j'-m) \\
=\ &(-1)^{j-j'-1}\sqrt{J_{1+}(J_{1+}+1)(2J_{1+}+1)(2j+1)}(2J_++1) \\
&\times(-1)^{2s}(2s+1)W(J_{1+}s1j;j'J_{1+})(-1)^{2J_++2s-2J_{2+}-j-j'} \\
&\times W(J_-J_+sJ_{1+};J_{2+}j)W(J_{1+}sJ_+J_-;j'J_{2+})(j-m,10;j'-m) \\
=\ &(-1)^{2J_+-2J_{2+}-2j'-1}\sqrt{J_{1+}(J_{1+}+1)(2J_{1+}+1)(2j+1)}(2J_++1) \\
&\times W(j'1J_-J_+;jJ_+)W(J_{1+}1J_{2+}J_+;J_{1+}J_+) \\
&\times(j-m,10;j'-m). \qquad\qquad (A\cdot 29)
\end{aligned}
$$

where (A·9) is used.

Similarly, the second term on the right side on (B.2.6) becomes by symmetry

$$
\begin{aligned}
&(-1)^{2J_+-2J_{1+}-2j'-1}\sqrt{J_{2+}(J_{2+}+1)(2J_{2+}+1)(2j+1)}(2J_++1) \\
&\times W(j'1J_-J_+;jJ_+)W(J_{2+}1J_{1+}J_+;J_{2+}J_+) \\
&\times(j-m,10;j'-m). \qquad\qquad (A\cdot 30)
\end{aligned}
$$

The sum of (A·29) and (A·30) together with the m term and the explicit forms

$$
W(J_{1+}1J_{2+}J_+;J_{1+}J_+) = \frac{J_{1+}(J_{1+}+1)+J_+(J_++1)-J_{2+}(J_{2+}+1)}{\sqrt{4J_{1+}(J_{1+}+1)(2J_{1+}+1)J_+(J_++1)(2J_++1)}},
$$

$$
W(J_{2+}1J_{1+}J_+;J_{2+}J_+) = (J_{1+}\leftrightarrow J_{2+}\text{ from above}),
$$

give the term of (A·26) and it follows that the vector addition coefficient is given by (A·25).

References

1) T. Maekawa, J. Math. Phys. **20** (1978), 691. Related papers are cited there.
2) G. Racah, Phy. Rev. **62** (1942), 438.