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Topological pressure of Cantor minimal systems within a strong orbit equivalence class

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Abstract.
In this paper we will show that for any Cantor minimal system (X, ϕ), any potential function f and any c with $\sup\{\int f d\mu \mid \mu$ is a ϕ-invariant probability measure on $X\} \leq c \leq \infty$, there exists a Cantor minimal system (Y, ψ) such that ϕ and ψ are strongly orbit equivalent and the topological pressure of ψ determined by f is equal to c. If c is finite, we can take ψ as a (minimal) subshift. This result is generalization of the paper [S3]: On the subshift within strong orbit equivalence class for minimal homeomorphisms.

1. Introduction

Let X be a Cantor set and $T : X \to X$ be a homeomorphism acting minimally (i.e. for any $x \in X$, the orbit $\{T^n x \mid n \in \mathbb{Z}\}$ is dense in X). A pair (X, T) is called *a Cantor minimal system*. Giordano, Putnam and Skau showed that the following statements are equivalent ([GPS]: Theorem 2.1):

- Two Cantor minimal systems are strongly orbit equivalent.
- Two C^*-crossed products associated with Cantor minimal systems are isomorphic.

This theorem is the topological/C^*-algebra setting of Krieger's theorem ([Kr1], [Kr2]). (The measure-theoretic/von Neumann algebra setting means a relationship between the measure-theoretic orbit equivalence of ergodic non-singular systems and an isomorphism of von Neumann crossed product factors.) In the measure-theoretic setting, Dyes showed any ergodic measure preserving systems are orbit equivalent ([Dy1], [Dy2]). It is not hard to construct an ergodic measure preserving system having any fixed value of measure-theoretic entropy. These imply that the concepts of orbit equivalence and measure-theoretic entropy are independent. In

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the topological setting we have a similar result, that is, the concepts of strong orbit equivalence and topological entropy are independent ([S1],[S2]).

Concerning the strong orbit equivalence of a Cantor minimal system, we can ask the following question. Within any strong orbit equivalence class, is there a minimal subshift? In [S3], we obtain the following result: For every Cantor minimal system its strong orbit equivalence class contains minimal subshift of all finite topological entropies. In this paper we generalize this result using the concept of topological pressure by the following (Theorem 1.1). For a topological dynamical system (X,T), denote $\mathcal{M}(X)$ by the set of Borel probability measures on X and $\mathcal{M}(X,T)$ by the set of T-invariant Borel probability measures on X. Let $C(X,\mathbb{R})$ denote the set of all real valued continuous functions.

Theorem 1.1 *Suppose that (X,ϕ) is a Cantor minimal system and $f \in C(X,\mathbb{R})$, which is called a potential function, is given. Choose any α with*

$$\exp\left(\sup\left\{\int f d\mu \;\middle|\; \mu \in \mathcal{M}(X,\phi)\right\}\right) \leq \alpha \leq \infty \tag{1.1}$$

and fix it. Then there exists a Cantor minimal system (Y,ψ) strongly orbit equivalent to (X,ϕ) such that

$$P(\psi, f\circ\theta^{-1}) = \log\alpha,$$

where $P(\psi,\cdot)$ is the topological pressure of ψ and $\theta : X \to Y$ is a strong orbit equivalence map. If α is finite, we can take ψ as a minimal subshift.

We remark that if $f \equiv 0$, then $1 \leq \alpha \leq \infty$ and $P(\psi,0)$ is the topological entropy of ψ. So Theorem 1.1 is the generalization of [S1], [S2] and [S3]. We also remark that (1.1) is the best possible inequality which α can take. The reason is the following. Giordano, Putnam and Skau showed that an (strong) orbit equivalence map $\theta : X \to Y$ gives a bijection $\tilde{\theta} : \mathcal{M}(Y,\psi) \to \mathcal{M}(X,\phi)$ defined by $\tilde{\theta}(\nu) = \nu\circ\theta$ (Theorem 2.2 in [GPS]). Using this fact and the variational principle of topological pressure (see Theorem 9.10 in [W1]), we have

$$\begin{aligned} P(\psi, f\circ\theta^{-1}) &= \sup\left\{h_\nu(\psi) + \int f\circ\theta^{-1} d\nu \;\middle|\; \nu \in \mathcal{M}(Y,\psi)\right\} \\ &\geq \sup\left\{\int f\circ\theta^{-1} d\nu \;\middle|\; \nu \in \mathcal{M}(Y,\psi)\right\} \\ &= \sup\left\{\int f d\tilde{\theta}(\nu) \;\middle|\; \nu \in \mathcal{M}(Y,\psi)\right\} \\ &= \sup\left\{\int f d\mu \;\middle|\; \mu \in \mathcal{M}(X,\phi)\right\}. \end{aligned}$$

Now we give an overview of each section below. In this section below we introduce some notations, definitions and conditions concerning Bratteli diagrams. In §2, we consider the relation between Cantor minimal systems and subshifts. We will show that whenever a properly ordered Bratteli diagram $\tilde{\mathcal{B}}$ satisfies Property 1.5, then the associated Bratteli-Vershik system $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is topologically conjugate to a subshift (Theorem 2.4). In §3, we calculate a topological pressure of a special case of Cantor minimal system. By Theorem 3.8, we only calculate a pressure of a subshift associated with a $\tilde{\mathcal{B}}$ satisfying Property 1.5. In §4, we introduce two modification propositions of diagram which preserve the equivalence relation on Bratteli diagrams. In Proposition 4.2 we construct a based Bratteli diagram $\mathcal{C}$ using a given diagram $\mathcal{B}$. In Proposition 4.5 we construct the desired diagram $\tilde{\mathcal{B}}$ of (Y, ψ) in Theorem 1.1 using a based diagram $\mathcal{C}$. These propositions play important roles in proving Theorem 1.1. Finally in §5, we prove Theorem 1.1.

Notation 1.2 Basically, we use notations and definitions in [HPS] and [GPS]. Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered (also called simply ordered) Bratteli diagram. Suppose A is a set and $|A|$ (or $\#A$) denotes the cardinality of A.

(1) Let $r : E \to V$ denote the range map and $s : E \to V$ denote the source map. Namely, $e \in E_n$ connects between $s(e) \in V_{n-1}$ and $r(e) \in V_n$.

(2) Let $M^{(n)} = [\#r^{-1}(u) \cap s^{-1}(v)]_{u \in V_n, v \in V_{n-1}}$ denote the n-th incidence matrix of $\mathcal{B}$ (i.e., $M^{(n)}_{uv}$ is the number of edges connecting between $u \in V_n$ and $v \in V_{n-1}$). We also write $\mathcal{B} = (V, E, \{M^{(n)}\}, \geq)$. Let $M^{(n)}_u = [M^{(n)}_{uv}]_{v \in V_{n-1}}$ denote the u's row vector of $M^{(n)}$ which is called *an incidence vector of* u. For $n \geq k$, let $M^{(n,k)}$ denote the product of incidence matrices $M^{(n)} M^{(n-1)} \ldots M^{(k)}$.

(3) Set $X_{\mathcal{B}} = \{(e_i)_{i \in \mathbb{N}} \mid e_i \in E_i, r(e_i) = s(e_{i+1}) \ \forall i \in \mathbb{N}\}$. We call it *the (infinite lengths) path space of* $\mathcal{B}$. For $v \in V_n$, let $\mathcal{P}(v)$ denote the set of all (finite lengths) paths connecting between the top vertex $v_0 \in V_0$ and v. Then $|\mathcal{P}(v)| = M^{(n,1)}_{v v_0}$ holds. Put $\mathcal{P}(V_n) = \cup_{v \in V_n} \mathcal{P}(v)$. The range map r is extended to $\mathcal{P}(V_n)$, that is, for $p = (e_1, \ldots, e_n) \in \mathcal{P}(V_n)$, we define $r(p) = r(e_n)$.

(4) For $x = (e_i)_{i \in \mathbb{N}} \in X_{\mathcal{B}}$ or $x = (e_1, \ldots, e_n) \in \mathcal{P}(V_n)$, put $x_{[i,j]} = (e_i, e_{i+1}, \ldots, e_j)$ and $x_{(i,j]} = (e_{i+1}, \ldots, e_j)$. For $p \in \mathcal{P}(V_n)$, set $[p]_{\mathcal{B}} = \{x \in X_{\mathcal{B}} \mid x_{[1,n]} = p\}$. We call it *the cylinder set of* p.

(5) For $v \in V_n$ and $e \in r^{-1}(v)$, let Order(e) denote the order of e in $r^{-1}(v)$. If $p_{\min} = (e_1, e_2, \cdots)$ is the unique minimal path in $X_{\mathcal{B}}$, then Order$(e_n) = 1$ for all $n \in \mathbb{N}$. If $p_{\max} = (f_1, f_2, \cdots)$ is the unique maximal path in $X_{\mathcal{B}}$, then Order$(f_n) = |r^{-1}r(f_n)|$ for all $n \in \mathbb{N}$. Similarly, Order$(\cdot)$ is defined on $\mathcal{P}(V_n)$. I.e., for $p \in \mathcal{P}(V_n)$, Order(p) is the order of p in $\mathcal{P}(r(p))$.

(6) For $v \in V_n$, we write $r^{-1}(v) = \{e_i \mid 1 \leq i \leq |r^{-1}(v)|, \text{ Order}(e_i) = i\}$. Define

$$\text{List}(v) = (s(e_1), s(e_2), \cdots, s(e_{|r^{-1}(v)|})) \in (V_{n-1})^{|r^{-1}(v)|}.$$

We call it *the order list of* v.

(7) For a monotone increasing sequence $\{t_n\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ with $t_0 = 0$, we say that a Bratteli diagram $\mathcal{B}' = (V', E', \{M'^{(n)}\})$ is a *telescoping* (or *contraction*) of $\mathcal{B}$ to $\{t_n\}$, which we write $\mathcal{B}' = (\mathcal{B}, \{t_n\})$, if $V'_n = V_{t_n}$ and $M'^{(n)} = M^{(t_n, t_{n-1}+1)}$. Let $E_{t_n, t_{n-1}+1} = \{x_{(t_{n-1}, t_n]} \mid x \in X_{\mathcal{B}}\}$. Then there is a bijection between E'_n and $E_{t_n, t_{n-1}+1}$ preserving source and range vertices. We call $\{t_n\}$ a *sequence of telescoping depths*. Especially, we define $\mathcal{B}_{\text{odd}}$ as telescoping $\mathcal{B}$ to odd depths $\{0, 1, 3, \cdots\}$ and define $\mathcal{B}_{\text{even}}$ as telescoping $\mathcal{B}$ to even depths $\{0, 2, 4, \cdots\}$.

(8) Let $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ denote the Bratteli-Vershik system of $\mathcal{B}$. Namely, $\lambda_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$ is a lexicographic transformation defined by the order $\geq$ on E.

(9) For Bratteli diagrams $\mathcal{B}$ and $\mathcal{B}'$, define $\mathcal{B} \sim \mathcal{B}'$ provided that there exists a Bratteli diagram $\tilde{\mathcal{B}}$ such that $\tilde{\mathcal{B}}_{\text{odd}}$ yields a telescoping either $\mathcal{B}$ or $\mathcal{B}'$, and $\tilde{\mathcal{B}}_{\text{even}}$ yields a telescoping of the other. Then it is not hard to show that $\sim$ is an equivalence relation on Bratteli diagrams.

Remark 1.3

(1) Let (X, T) denote a Cantor minimal system, $C(X, \mathbb{Z})$ the set of all integer valued continuous functions, $C(X, \mathbb{Z})^+ = \{f \in C(X, \mathbb{Z}) \mid f \geq 0\}$ and $B_T = \{f - f \circ T^{-1} \mid f \in C(X, \mathbb{Z})\}$. Define

$$K^0(X, T) = C(X, \mathbb{Z})/B_T, \quad K^0(X, T)^+ = C(X, \mathbb{Z})^+/B_T.$$

In [Pu], Putnam showed that the triple $(K^0(X,T), K^0(X,T)^+, [1])$ is a simple, acyclic (i.e. $K^0(X,T) \ncong \mathbb{Z}$) dimension group with the (canonical distinguished) order unit [1], where $1 = 1_X$ is the constant function 1. Herman, Putnam and Skau showed in [HPS] that the family of Cantor minimal systems coincides with the family of Bratteli-Vershik systems up to conjugacy and showed that $K^0(X,T) \cong K_0(V, E)$ ($\cong$ means two dimension groups are unital order isomorphic), where (V, E) is a Bratteli-Vershik representation of (X, T) and $K_0(V, E)$ is defined by the induct limit of a system of ordered groups

$$K_0(V, E) = \lim_{n\to\infty} (\mathbb{Z}^{|V_{n-1}|}, M_n) = \mathbb{Z}^{|V_0|} \xrightarrow{M_1} \mathbb{Z}^{|V_1|} \xrightarrow{M_2} \mathbb{Z}^{|V_2|} \xrightarrow{M_3} \cdots.$$

They also showed that all (acyclic) simple dimension groups can be obtained in this (dynamical) way.

(2) It is easy to see that $(V, E) \sim (V', E')$ if and only if $K_0(V, E) \cong K_0(V', E)$. Giordano, Putnam and Skau showed in [GPS] that Bratteli-Vershik systems $(X_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ and $(X_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ are strongly orbit equivalent if and only if $\mathcal{B}_1 \sim \mathcal{B}_2$.

Definition 1.4

(1) (distinct order list.) We say V_n has *distinct order lists* if for $v, v' \in V_n$, $\mathrm{List}(v) = \mathrm{List}(v')$ implies $v = v'$ (or equivalently, $v \neq v'$ implies $\mathrm{List}(v) \neq \mathrm{List}(v')$).

(2) (The minimal/maximal vertex property.) Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered Bratteli diagram. We say E_n has *the minimal/maximal vertex property* if there exist $v_{\min}^{n-1}, v_{\max}^{n-1} \in V_{n-1}$ such that for any $e, f \in E_n$ with $\mathrm{Order}(e) = 1$ and $\mathrm{Order}(f) = |r^{-1}r(f)|$, then $s(e) = v_{\min}^{n-1}$ and $s(f) = v_{\max}^{n-1}$.

Now we consider a properly ordered Bratteli diagram $\tilde{\mathcal{B}}$ of Property 1.5. Later we will show that the associated Bratteli-Vershik system $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is conjugate to a subshift and its topological pressure is calculable.

Property 1.5 $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}, \tilde{\geq})$ satisfies the following properties. For any $n \in \mathbb{N}$,

(1) $\tilde{M}^{(n)}$ is a positive matrix (i.e. $\tilde{M}^{(n)}_{u,v} \geq 1$ for all u and v),

(2) $\tilde{E}_n$ has the minimal/maximal vertex property,

(3) $|\tilde{V}_n| \geq 3$ and $v_{\min}^n \neq v_{\max}^n$, where $v_{\min}^n$ and $v_{\max}^n$ are defined in Definition 1.4 (2),

(4) for each $v \in \tilde{V}_n$, $\tilde{M}^{(n)}_{vv_{\min}^{n-1}} = \tilde{M}^{(n)}_{vv_{\max}^{n-1}} = 1$,

(5) $\tilde{V}_n$ has distinct order lists. (In the case of $n = 1$, we ignore this property.)

2. Conjugacy between Cantor minimal systems and subshifts

In this section we consider a $\tilde{\mathcal{B}}$ satisfying Property 1.5. We will show that $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$ is topologically conjugate to a subshift. The details of shift spaces and its topology, see [LM] in §1 and §6.

Definition 2.1

(1) Let (X, σ) denote a subshift, that is, X is a shift space and σ is shift transformation. For $x \in X$ and $i, j \in \mathbb{Z}$ with $i \geq j$, set

$$x_{[i,j]} = x_i x_{i+1} \cdots x_j, \quad x_{[i,j)} = x_i x_{i+1} \cdots x_{j-1},$$

which are called *blocks* (*or words*) of x. Set

$$B_n(X) = \{x_{[0,n)} \mid x \in X\}, \quad B(X) = \cup_{n \in \mathbb{N}} B_n(X).$$

Since X is shift invariant, we see that $B_n(X) = \{x_{[i,j)} \mid x \in X,\ j - i = n\}$ and hence $B_n(X)$ is the set of all (length) n-blocks that occur in points in X. We call $B(X)$ *the language of* X. For $B \in B_n(X)$ and i, j with $j - i + 1 = n$, put

$$[B]_i^j = \{x \in X \mid x_{[i,j]} = B\}.$$

(2) (shift of finite type) Let A be an alphabet (a finite set) and $\mathcal{F}$ be a set of words with alphabet A. For $\mathcal{F}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $A^{\mathbf{Z}}$ which do not contain any word in $\mathcal{F}$. We say a subshift (X, σ) is *shift of finite type* (*SFT*) if X has the form $X_{\mathcal{F}}$ for some $\mathcal{F}$ and $\mathcal{F}$ is a finite set. We say an SFT subshift $(X_{\mathcal{F}}, \sigma)$ is *M-step* ($M \in \mathbb{N}$) if $\mathcal{F}$ consists of blocks with length $M+1$. We say a subshift (X, σ) is *irreducible* if for any $u, w \in B(X)$, there exists $v \in B(X)$ such that $uvw \in B(X)$.

Remark 2.2 Suppose $X \subset A^{\mathbf{Z}}$. By Theorem 6.1.21 in [LM],

X is a shift space iff $\exists \mathcal{F}$ such that $X = X_{\mathcal{F}}$ iff X is shift-invariant and compact.

Definition 2.3 (Subshift associated with $\tilde{B}$) Suppose $\tilde{B} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram. Let $\tau : X_{\tilde{B}} \cup (\cup_{i \in \mathbf{N}} \mathcal{P}(\tilde{V}_i)) \to \mathcal{P}(\tilde{V}_1)$ denote a truncation map, that is, $\tau x = x_1$ where $x = (x_1, x_2, \cdots)$.

(1) Define a shift invariant subset $X_\infty \subset \mathcal{P}(\tilde{V}_1)^{\mathbf{Z}}$ to be

$$X_\infty = \left\{ (\tau \lambda_{\tilde{B}}{}^n x)_{n \in \mathbf{Z}} \mid x \in X_{\tilde{B}} \right\}.$$

One can show that X_∞ is compact. Let σ_∞ denote the restriction of shift to X_∞.

(2) Define a finite directed graph $G_k = (\mathcal{V}, \mathcal{E})$ arising from $\mathcal{P}(\tilde{V}_k)$ as follows. Define a edge set $\mathcal{E} = \mathcal{P}(\tilde{V}_k)$ and a vertex set $\mathcal{V} = \{i(p), t(p) \mid p \in \mathcal{E}\}$, where $i(p)$ ($t(p)$, resp.) is the initial (terminal, resp.) vertex of p satisfying that

$$p, q \in \mathcal{E},\ t(p) = i(q) \text{ iff } \begin{cases} r(p) = r(q) \in \tilde{V}_k \text{ and } \mathrm{Order}(p) + 1 = \mathrm{Order}(q), \\ \text{or} \\ \mathrm{Order}(p) = |\mathcal{P}(r(p))| \text{ and } \mathrm{Order}(q) = 1. \end{cases}$$

It is easy to see that G_k is a irreducible graph. Let $\tilde{X}_k$ denote the edge shift X_{G_k}. I.e.,

$$\tilde{X}_k = X_{G_k} = \{x = (x_i)_{i \in \mathbf{Z}} \in \mathcal{P}(\tilde{V}_k)^{\mathbf{Z}} \mid t(x_i) = i(x_{i+1}) \text{ for all } i \in \mathbb{Z}\}.$$

(See [LM]:Definition 2.2.5.) Let $\tilde{\sigma}_k$ denote the shift on $\tilde{X}_k$. It is easy to see that $(\tilde{X}_k, \tilde{\sigma}_k)$ is a 1-step shift of finite type. Define $X_k = \pi_k(\tilde{X}_k)$, where the map $\pi_k : \tilde{X}_k \to \mathcal{P}(\tilde{V}_1)^{\mathbf{Z}}$ is defined by

$$\pi_k(\cdots x_{-1}.x_0 x_1 \cdots) = (\cdots(\tau x_{-1}).(\tau x_0)(\tau x_1)\cdots). \tag{2.1}$$

Let σ_k denote the shift on X_k.

First we consider the relationship between $(X_{\tilde{B}}, \lambda_{\tilde{B}})$ and $(X_\infty, \sigma_\infty)$.

Theorem 2.4 *Suppose $\tilde{B} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property* 1.5. *Then $(X_{\tilde{B}}, \lambda_{\tilde{B}})$ is topologically conjugate to $(X_\infty, \sigma_\infty)$.*

Proof. We write $\lambda = \lambda_{\tilde{B}}$ for short. Define $\pi_\infty : X_{\tilde{B}} \to X_\infty$ as

$$\pi_\infty x = (\tau \lambda^n x)_{n \in \mathbb{Z}}.$$

We will show that π_∞ is a conjugacy. Clearly π_∞ is surjective. $\pi_\infty \circ \lambda = \sigma_\infty \circ \pi_\infty$ holds because

$$(\pi_\infty \lambda x)_n = \tau \lambda^n \lambda x = \tau \lambda^{n+1} x = (\pi_\infty x)_{n+1} = (\sigma_\infty \pi_\infty x)_n.$$

Therefore we will show that π_∞ is injective. We call the argument below *the one-to-one argument.*

The one-to-one argument. Choose any $x = (x_i), y = (y_i) \in X_{\tilde{B}}$ with $x \neq y$ and fix them. It suffices to show that there is $m \in \mathbb{Z}$ so that $\tau\lambda^m x \neq \tau\lambda^m y$. If $\tau x \neq \tau y$, the claim would have been proven. Therefore assume that there is $l > 1$ so that $x_{[1,l]} = y_{[1,l]}$ and $x_{[1,l+1]} \neq y_{[1,l+1]}$ ($x_{[1,l]} = (x_1, x_2, \ldots, x_l)$). Suppose $n \leq 0$ is the maximum number so that $(\lambda^n x)_{[1,l+1]}$ lies in the minimal path in $\mathcal{P}(r(x_{l+1}))$. This implies that $\mathrm{Order}((\lambda^n x)_{l+1}) = 1$ and $(\lambda^n x)_{[l+2,\infty)} = x_{[l+2,\infty)}$. Then we consider the following two cases:

(i) $(\lambda^n x)_{[1,l]} = (\lambda^n y)_{[1,l]}$,

(ii) $(\lambda^n x)_{[1,l]} \neq (\lambda^n y)_{[1,l]}$.

In the case of (i), we note that $\mathrm{Order}((\lambda^n y)_{l+1}) = 1$ because $r(\lambda^n x)_l = r(\lambda^n y)_l = v^l_{\min}$ and Property 1.5 (2) and (4). Let $u = r(\lambda^n x)_{l+1}$ and $v = r(\lambda^n y)_{l+1}$. Then $u \neq v$ because of $x_{[1,l+1]} \neq y_{[1,l+1]}$. Since $\tilde{V}_{l+1}$ has distinct order lists, there exist $e \in r^{-1}(u)$, $f \in r^{-1}(v)$ and the minimum number $1 < n' \leq \min(|r^{-1}(u)|, |r^{-1}(v)|)$ such that $s(e) \neq s(f)$ and $\mathrm{Order}(e) = \mathrm{Order}(f) = n'$. Let $\tilde{n} = n + \sum_{i=1}^{n'-1} |\mathcal{P}(s(e_i))|$, where $e_i \in r^{-1}(u)$ with $\mathrm{Order}(e_i) = i$. Then $s(\lambda^{\tilde{n}} x)_{l+1} = s(e)$ and $s(\lambda^{\tilde{n}} y)_{l+1} = s(f)$. This implies that $(\lambda^{\tilde{n}} x)_{[1,l]} \neq (\lambda^{\tilde{n}} y)_{[1,l]}$.

Both the case (i) and (ii) imply that there exists $N \in \mathbb{Z}$ such that $(\lambda^N x)_{[1,l]} \neq (\lambda^N y)_{[1,l]}$ holds. By repeating this procedure, we get $\tau\lambda^m x \neq \tau\lambda^m y$ for some $m \in \mathbb{Z}$. So we finish the proof.

Definition 2.5 For $v \in V \setminus V_0$, define words (or blocks) $\mathrm{Con}(v)$ and $\tau\mathrm{Con}(v)$ as

$$\mathrm{Con}(v) = p_1 p_2 \ldots p_{|\mathcal{P}(v)|}, \quad \tau\mathrm{Con}(v) = (\tau p_1)(\tau p_2) \ldots (\tau p_{|\mathcal{P}(v)|}),$$

where $\{p_i \mid \mathrm{Order}(p_i) = i,\ 1 \leq i \leq |\mathcal{P}(v)|\} = \mathcal{P}(v)$.

Remark 2.6 Using Con(·) and τCon(·), we see that

$$\tilde{X}_k = \left\{ x \in \mathcal{P}(\tilde{V}_k)^{\mathbb{Z}} \;\middle|\; \begin{array}{c} \exists\{n_i\}_{i\in\mathbb{Z}} \subset \mathbb{Z} \text{ with } n_i < n_{i+1},\ \exists\{v_i\}_{i\in\mathbb{Z}} \\ \subset \tilde{V}_k \text{ s.t.} \forall i \in \mathbb{Z},\ x_{[n_i,n_{i+1})} = \mathrm{Con}(v_i) \end{array} \right\},$$

$$X_k = \left\{ x \in \mathcal{P}(\tilde{V}_1)^{\mathbb{Z}} \;\middle|\; \begin{array}{c} \exists\{n_i\}_{i\in\mathbb{Z}} \subset \mathbb{Z} \text{ with } n_i < n_{i+1},\ \exists\{v_i\}_{i\in\mathbb{Z}} \\ \subset \tilde{V}_k \text{ s.t. } \forall i \in \mathbb{Z},\ x_{[n_i,n_{i+1})} = \tau\mathrm{Con}(v_i) \end{array} \right\}.$$

So $(\tilde{X}_k, \tilde{\sigma}_k)$ and (X_k, σ_k) are renewal systems with the generating list $\{\mathrm{Con}(v) \mid v \in \tilde{V}_k\}$, $\{\tau\mathrm{Con}(v) \mid v \in \tilde{V}_k\}$ respectively (see [LM], §13.1).

We consider the relationship between $(\tilde{X}_k, \tilde{\sigma}_k)$ and (X_k, σ_k). The following theorem is important so as to calculate the topological pressure of $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$

Theorem 2.7 *Suppose $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property* 1.5. *Then for any $k \in \mathbb{N}$, $(\tilde{X}_k, \tilde{\sigma}_k)$ and (X_k, σ_k) are topologically conjugate.*

Proof. We will show that the map π_k is a conjugacy. Clearly π_k is surjective and $\pi_k \circ \tilde{\sigma}_k = \sigma_k \circ \pi_k$. So we will show that π_k is injective. Suppose $x = (x_i), x' = (x'_i) \in \tilde{X}_k$ satisfies that $x \neq x'$ and x_0 is some minimal path in $\mathcal{P}(\tilde{V}_k)$. If $\tau x_0 \neq \tau x'_0$, then we have been done. Therefore we assume $\tau x_0 = \tau x'_0$. Then there exist $\{n_i\}, \{n'_i\} \subset \mathbb{Z}$ and $\{v_i\}, \{v'_i\} \subset \tilde{V}_k$ such that for any $i \in \mathbb{Z}$,

$$x_{[n_i,n_{i+1})} = \mathrm{Con}(v_i),\ x'_{[n'_i,n'_{i+1})} = \mathrm{Con}(v'_i), n_0 = 0,\ n'_0 \leq 0 < n'_1.$$

Here, let us consider the following three cases:

(i) $v'_0 \neq v_0$,

(ii) $v'_0 = v_0$ and $n'_0 \neq 0$,

(iii) $v'_0 = v_0$ and $n'_0 = 0$.

In the case of (i) and (ii), there exists l with $1 \leq l < k$ such that $(x_0)_{[1,l]} = (x'_0)_{[1,l]}$ and $(x_0)_{[1,l+1]} \neq (x'_0)_{[1,l+1]}$. So we use the one-to-one argument in Theorem 2.4 and obtain $\tau x_m \neq \tau x'_m$ for some m. In the case of (iii), by $x \neq x'$ there exists $I \in \mathbb{N}$ such that

- for any i with $|i| < I$, $v_i = v'_i$ (therefore $n_i = n'_i$ holds),
- $v_I \neq v'_I$ or $v_{-I} \neq v'_{-I}$.

If $v_I \neq v'_I$, by Property 1.5 (2), $(x_{n_I})_{[1,k]}$ and $(x'_{n_I})_{[1,k]}$ are the minimal path in $\mathcal{P}(v_I)$ and $\mathcal{P}(v'_I)$ respectively (and hence $(x_{n_I})_{[1,k-1]} = (x'_{n_I})_{[1,k-1]}$) and $(x_{n_I})_{[1,k]} \neq (x'_{n_I})_{[1,k]}$. So using the one-to-one argument in Theorem 2.4, we have $\tau x_m \neq \tau x'_m$ for some m. If $v_{-I} \neq v'_{-I}$, basically by the same argument we have $\tau x_m \neq \tau x'_m$ for some m.

In the case where x_0 is not some minimal path, we may consider some minimal path x_n instead of x_0. Therefore we have a conclusion that π_k is injective.

3. Calculation of topological pressure

The aim of this section is to calculate the topological pressure of a Bratteli-Vershik system in a special case. First we introduce the definition of topological pressure. The details of definitions and notations are written in [W1].

3.1. Definitions and properties of topological pressure

Definition 3.1 Let (X,T) be a topological dynamical system. (I.e. X is a compact metric space and T is a continuous transformation on X.) For $f \in C(X,\mathbb{R})$ and $n \in \mathbb{N}$, put $(S_n f)(x) = \sum_{i=0}^{n-1} f(T^i x)$. For $\varepsilon > 0$, put

$$\begin{aligned} Q_n(T,f,\varepsilon) &= \inf\left\{ \sum_{x\in F} e^{(S_n f)(x)} \mid F \text{ is a } (n,\varepsilon)\text{-spanning set for } X \right\}, \\ Q(T,f,\varepsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T,f,\varepsilon), \\ P(T,f) &= \lim_{\varepsilon\to 0} Q(T,f,\varepsilon). \end{aligned}$$

Then it is easy to see that $P(T,f)$ exists but could be ∞. The map $P(T,\cdot) : C(X,\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is called the *topological pressure of* T.

When T is an expansive homeomorphism, we can calculate $P(T,f)$ as the following way. A finite open cover α of X is a *generator* for T if for every bisequence $\{A_n\}_{n=-\infty}^{\infty}$ of members of α, the set $\cap_{n=-\infty}^{\infty} T^{-n}\overline{A_n}$ contains at most one point of X. Define

$$p_n(T,f,\alpha) = \inf\left\{ \sum_{A\in\beta} \sup_{x\in A} e^{(S_n f)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} T^{-i}\alpha \right\}.$$

Theorem 3.2 ([W1]: Lemma 9.3, Theorem 9.6) *Let T be an expansive homeomorphism of X. If α is a generator for T, then*

$$P(T,f) = \lim_{n\to\infty} \frac{1}{n} \log p_n(T,f,\alpha) = \inf_{N\in\mathbb{N}} \frac{1}{N} \log p_N(T,f,\alpha).$$

In the case of a subshift (X,σ) with alphabet A, $\alpha = \{[a]_0^0 \mid a \in A\}$ is generator for σ. Moreover we see that

- $\vee_{i=0}^{n-1}\sigma^{-i}\alpha = \{[B]_0^{n-1} \mid B \in B_n(X)\}$ and hence $\vee_{i=0}^{n-1}\sigma^{-i}\alpha$ is a finite cover of X,
- Since $\{[B]_0^{n-1} \mid B \in B_n(X)\}$ is a disjoint finite cover (i.e., $B \neq B'$ implies $[B]_0^{n-1} \cap [B']_0^{n-1} = \emptyset$), it has no proper subcover.

So by Theorem 3.2 we have the following.

Proposition 3.3 *Suppose that (X,σ) is a subshift and $f \in C(X,\mathbb{R})$ is potential function. Then*

$$P(\sigma,f) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{B\in B_n(X)}\sup_{x\in[B]_0^{n-1}} e^{(S_nf)(x)}\right)$$
$$= \inf_{N\in\mathbb{N}}\frac{1}{N}\log\left(\sum_{B\in B_N(X)}\sup_{x\in[B]_0^{N-1}} e^{(S_Nf)(x)}\right).$$

3.2. Topological pressure of Bratteli-Vershik systems

In this subsection we assume that $\tilde{\mathcal{B}}$ satisfies Property 1.5. First we calculate the topological pressure of $(\tilde{X}_k,\tilde{\sigma}_k)$ with respect to some special potential functions.

Definition 3.4 Suppose $\mathcal{B}$ is a properly ordered Bratteli diagram. We say that f is a *simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_n)$* if for any $x,x'\in X_{\mathcal{B}}$ with $x_{[1,n]} = x'_{[1,n]}$, $f(x)=f(x')$ holds. Then for $p\in\mathcal{P}(V_n)$ we can define $f[p]_{\mathcal{B}} = f(x)$ if $x\in[p]_{\mathcal{B}}$.

Remark 3.5 Since each cylinder set $[p]_{\mathcal{B}}$ is a clopen set, f is a continuous function.

For $g\in C(X_{\tilde{\mathcal{B}}},\mathbb{R})$ and $k\in\mathbb{N}$, let g_k denote a simple function based on $\mathcal{P}(\tilde{V}_k)$ satisfying $\lim_{k\to\infty} g_k = g$ as the supremum norm. We define a continuous function $\tilde{g}_k$ on $\tilde{X}_k$ to be

$$\tilde{g}_k(x) = g_k[x_0]_{\tilde{\mathcal{B}}},$$

where $x=(x_n)\in\tilde{X}_k$ and hence $\tilde{g}_k$ is a simple function on $\tilde{X}_k$.

Lemma 3.6 *In the situation above, we have*

$$P(\tilde{\sigma}_k,\tilde{g}_k) = \log\alpha_k,$$

where α_k is the maximum positive solution of the equation for x given by

$$\sum_{v\in\tilde{V}_k}\frac{\Gamma(v)}{x^{|\mathcal{P}(v)|}} = 1, \quad \textit{where } \Gamma(v) = \exp\left(\sum_{p\in\mathcal{P}(v)} g_k[p]_{\tilde{\mathcal{B}}}\right).$$

Proof. By Theorem 2.7, $(\tilde{X}_k,\tilde{\sigma}_k)$ is 1-step irreducible SFT. Let A be the adjacency matrix of the graph G_k defined by

$$A_{pq} = \begin{cases} 1, & \text{if } t(p)=i(q),\\ 0, & \text{otherwise.}\end{cases}$$

Let D be a diagonal matrix defined by $D_{pp} = e^{g_k[p]_{\tilde{\mathcal{B}}}}$. Put $S=AD$. Let $\lambda_S = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } S\}$. A is an irreducible matrix and so is S. Then using Lemma 4.7 in [W2], we have $P(\tilde{\sigma}_k, \tilde{g}_k) = \log \lambda_S$ and there exists an eigenvalue λ such that $\lambda_S = \lambda$. Now we will show $\lambda_S = \alpha_k$. By Perron-Frebenius Theorem (See [W1]: p16, Theorem 0.16.), λ_S is an eigenvalue and its eigenvector is positive. Let θ be the right eigenvector of λ_S. We write θ as $\theta = (\theta_v) \in \mathbb{R}_+^{|\mathcal{P}(\tilde{V}_k)|}$, where $\theta_v = (\theta_p)_{p\in\mathcal{P}(v)}$. $(T - \lambda_S)\theta = 0$ follows that

- $-\lambda_S\theta_p + e^{g_k[q]_{\tilde{B}}}\theta_q = 0$, where $r(p) = r(q)$ and $\mathrm{Order}(p) + 1 = \mathrm{Order}(q)$.
- $-\lambda_S\theta_p + \sum_q e^{g_k[q]_{\tilde{B}}}\theta_q = 0$, where $\mathrm{Order}(p) = |\mathcal{P}(r(p))|$ and q is taken over $\mathrm{Order}(q) = 1$.

These are equivalent to

- $\theta_p = \lambda_S^{\mathrm{Order}(p)-1}\exp(-\sum_{p'} g_k[p']_{\tilde{B}})\theta_q$, where $r(p) = r(q)$, $\mathrm{Order}(q) = 1$ and p' is taken over $p' \in \mathcal{P}(r(p))$ with $1 < \mathrm{Order}(p') \le \mathrm{Order}(p)$.
- $\sum_q e^{g_k[q]_{\tilde{B}}}\theta_q = \lambda_S^{|\mathcal{P}(r(p))|}\exp(-\sum_{p'} g_k[p']_{\tilde{B}})\theta_p$, where $\mathrm{Order}(p) = 1$, q is taken over $\mathrm{Order}(q) = 1$ and p' is taken over $p' \in \mathcal{P}(r(p))$ with $1 < \mathrm{Order}(p') \le \mathrm{Order}(p)$.

Then we have

$$\theta_p = \frac{\lambda_S^{|\mathcal{P}(r(q))|}\exp(\sum_{p'} g_k[p']_{\tilde{B}})}{\lambda_S^{|\mathcal{P}(r(p))|}\exp(\sum_{q'} g_k[q']_{\tilde{B}})}\theta_q \Rightarrow \sum_{v\in\tilde{V}_k} \frac{\lambda_S^{|\mathcal{P}(r(q))|-|\mathcal{P}(v)|}\Gamma(v)}{\exp(\sum_{q'} g_k[q']_{\tilde{B}})}\theta_q = \frac{\lambda_S^{|\mathcal{P}(r(q))|}}{\exp(\sum_{q'} g_k[q']_{\tilde{B}})}\theta_q,$$

where p' is taken over $p' \in \mathcal{P}(r(p))$ with $1 < \mathrm{Order}(p') \le \mathrm{Order}(p)$ and q' is taken over $q' \in \mathcal{P}(r(q))$ with $1 < \mathrm{Order}(q') \le \mathrm{Order}(q)$. So we have

$$\sum_{v\in\tilde{V}_k} \frac{\Gamma(v)}{\lambda_S^{|\mathcal{P}(v)|}} = 1.$$

So we finish the proof.

Lemma 3.7

$$P(\sigma_\infty, g\circ\pi_\infty^{-1}) = \lim_{k\to\infty} P(\sigma_k, \tilde{g}_k\circ\pi_k^{-1}). \tag{3.1}$$

Proof. First we will show

$$\bigcap_{k\in\mathbb{N}} X_k = X_\infty.$$

For $k \in \mathbb{N}$ and $v \in \tilde{V}_{k+1}$, the word $\tau\mathrm{Con}(v)$ corresponds to concatenated words

$$\tau\mathrm{Con}(u_1)\tau\mathrm{Con}(u_2)\cdots\tau\mathrm{Con}(u_n),$$

where $(u_1, u_2, \ldots, u_n) = \mathrm{List}(v)$. Then by Remark 2.6 $X_1 \supset X_2 \supset \cdots$ and $X_\infty \subset \cap_{k\in\mathbb{N}} X_k$. Conversely, suppose $x \in \cap_{k\in\mathbb{N}} X_k$. Since

$x \in X_\infty$ iff for any $n \in \mathbb{N}$, the word $x_{[-n,n]}$ appears in a point of X_∞, we will show $x_{[-n,n]} \in B_{2n+1}(X_\infty)$. It suffices to show that $x_{[-n,n]}$ appears in $\tau\mathrm{Con}(v)$ for some vertex v. Suppose that $N \in \mathbb{N}$ satisfies $\min\{|\mathcal{P}(v)| \mid v \in \tilde{V}_N\} > 2n$. For $m \geq N$, Define A_m, B_m and C_m as

$$A_m = \{v \in \tilde{V}_m \mid x_{[-n,n]} \text{ appears in } \tau\mathrm{Con}(v)\},$$
$$B_m = \{(u,v) \in \tilde{V}_m \times \tilde{V}_m \mid x_{[-n,n]} \text{ appears in a concatenated word } \tau\mathrm{Con}(u)\tau\mathrm{Con}(v)\},$$
$$C_m = \{(u,v) \in \tilde{V}_m \times \tilde{V}_m \mid \tau\mathrm{Con}(u)\tau\mathrm{Con}(v) \in B(X_{m+1})\}.$$

Since $x \in \cap_{k\in\mathbb{N}} X_k$, $B_m \cap C_m \neq \emptyset$ holds for any $m \geq N$. Suppose $A_m = \emptyset$ for any m. If $(u,v) \in B_N \cap C_N$ with $(u,v) \neq (v^N_{\max}, v^N_{\min})$, then there exist $w \in \tilde{V}_{N+1}$ and $e, f \in r^{-1}(w)$ such that $s(e) = u$, $s(f) = v$ and $\mathrm{Order}(f) = \mathrm{Order}(e) + 1$. But this implies that $w \in A_{N+1}$ and hence $B_N \cap C_N = \{(v^N_{\max}, v^N_{\min})\}$. Now, for any $y \in \tilde{V}_{N+2}$, $\tau\mathrm{Con}(y)$ contains the word $\tau\mathrm{Con}(v^N_{\max})\tau\mathrm{Con}(v^N_{\min})$. Because by Property 1.5 (2), any concatenated word $\tau\mathrm{Con}(u)\tau\mathrm{Con}(v)$ with $u, v \in \tilde{V}_{N+1}$ contains $\tau\mathrm{Con}(v^N_{\max})\tau\mathrm{Con}(v^N_{\min})$ and $\tau\mathrm{Con}(y)$ consists of concatenations of $\tau\mathrm{Con}(w)$'s ($w \in \tilde{V}_{N+1}$). Therefore $y \in A_{N+2}$ holds and hence it is a contradiction. Therefore $A_m \neq \emptyset$ for some m and $x_{[-n,n]}$ appears in $\tau\mathrm{Con}(v)$ for some vertex v.

Define $h_n \in C(X_{\tilde{\mathcal{B}}}, \mathbb{R})$ based on $\mathcal{P}(\tilde{V}_n)$ to be

$$h_n(x) = \max\{g(y) \mid y \in [p]_{\tilde{\mathcal{B}}}\} \quad \text{if } x \in [p]_{\tilde{\mathcal{B}}}.$$

Then we see that $\lim_{n\to\infty} \|h_n - g\| = 0$ and hence $\lim_{k\to\infty} \|h_n - g_n\| = 0$. By Theorem 9.7 (iv) in [W1]

$$|P(\sigma_k, \tilde{g}_k \circ \pi_k^{-1}) - P(\sigma_k, \tilde{h}_k \circ \pi_k^{-1})| \leq \|h_k - g_k\| \to 0 \quad \text{as } k \to \infty.$$

Therefore we will show

$$P(\sigma_\infty, g \circ \pi_\infty^{-1}) = \lim_{k\to\infty} P(\sigma_k, \tilde{h}_k \circ \pi_k^{-1}).$$

Clearly

$$P(\sigma_1, \tilde{h}_1) \geq P(\sigma_2, \tilde{h}_2 \circ \pi_2^{-1}) \geq P(\sigma_3, \tilde{h}_3 \circ \pi_3^{-1}) \geq \cdots \geq P(\sigma_\infty, g \circ \pi_\infty^{-1})$$

because $X_1 \supset X_2 \supset \cdots \supset X_\infty$ and h_n is monotone decreasing with respect to n. By Proposition 3.3 for any $\varepsilon > 0$, choose N satisfying

$$\frac{1}{N} \log \left(\sum_{B \in B_N(X_\infty)} \sup_{x \in [B]_0^{N-1}} e^{(S_N g \circ \pi_\infty^{-1})(x)} \right) < P(\sigma_\infty, g \circ \pi_\infty^{-1}) + \frac{1}{2}\varepsilon.$$

By $\cap_{k\in\mathbb{N}} X_k = X_\infty$, there exists $K \in \mathbb{N}$ such that for any $k \geq K$ and $x \in X_{\tilde{\mathcal{B}}}$,

$$B_N(X_\infty) = B_N(X_k) \quad \text{and} \quad h_k(x) < g(x) + \frac{1}{2}\varepsilon.$$

Using Proposition 3.3 again, for any $k \geq K$ we have

$$\begin{aligned} P(\sigma_k, \tilde{h}_k \circ \pi_k^{-1}) &\leq \frac{1}{N} \log \left(\sum_{B \in B_N(X_k)} \sup_{x \in [B]_0^{N-1}} e^{(S_N \tilde{h}_k \circ \pi_k^{-1})(x)} \right) \\ &< \frac{1}{N} \log \left(\sum_{B \in B_N(X_\infty)} \sup_{x \in [B]_0^{N-1}} e^{(S_N g \circ \pi_\infty^{-1})(x) + N\varepsilon/2} \right) \\ &< P(\sigma_\infty, g \circ \pi_\infty^{-1}) + \varepsilon. \end{aligned}$$

Theorem 3.8 *Suppose that $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \tilde{\geq})$ is a properly ordered Bratteli diagram satisfying Property* 1.5, *g is a potential function on $X_{\tilde{\mathcal{B}}}$ and $\{g_n\}$ is a sequence of simple functions on $X_{\tilde{\mathcal{B}}}$ based on $\mathcal{P}(\tilde{V}_n)$ for each n satisfying* $\lim_{n\to\infty} \|g - g_n\| = 0$. *Suppose α_n is the unique positive solution of the equation for x given by*

$$\sum_{v \in \tilde{V}_n} \frac{\Gamma_n(v)}{x^{|\mathcal{P}(v)|}} = 1, \quad \text{where } \Gamma_n(v) = \exp\left(\sum_{p \in \mathcal{P}(v)} g_n[p]_{\tilde{\mathcal{B}}} \right)$$

and $\lim_{n\to\infty} \alpha_n = \alpha$ *exists. Then* $P(\lambda_{\tilde{\mathcal{B}}}, g) = \log \alpha$.

Proof. By Theorem 2.4, $\lambda_{\tilde{\mathcal{B}}}$ and σ_∞ are conjugate and hence $P(\lambda_{\tilde{\mathcal{B}}}, g) = P(\sigma_\infty, g \circ \pi_\infty^{-1})$. By Theorem 2.7, $\tilde{\sigma}_k$ and σ_k are conjugate and hence $P(\tilde{\sigma}_k, \tilde{g}_k) = P(\sigma_k, \tilde{g}_k \circ \pi_k^{-1})$. Therefore by Lemma 3.6 and 3.7 we have

$$P(\lambda_{\tilde{\mathcal{B}}}, g) = \lim_{k\to\infty} P(\sigma_k, \tilde{g}_k \circ \pi_k^{-1}) = \lim_{k\to\infty} P(\tilde{\sigma}_k, \tilde{g}_k) = \lim_{k\to\infty} \log \alpha_k = \log \alpha.$$

4. The modification of simple Bratteli diagram preserving equivalence relation

In this section, we give two modifications of diagrams preserving the equivalence relation of Bratteli diagrams (see Notation 1.2 (9)). The first modification is useful for the construction of a based diagram $\mathcal{C}$ in the main theorem. Using a given simple Bratteli diagram $\mathcal{B} = (V, E, \{M^{(n)}\})$ and a sequence of telescoping depths $\{t_n\}_{n \in \mathbb{Z}_+}$, $\mathcal{C} = (W, F, \{N^{(n)}\})$ is constructed by the following: (We call the construction below *the vertex amalgamation.*)

The vertex amalgamation construction of $\mathcal{C}$. Define an equivalence relation $\sim$ on vertices of $(\mathcal{B}, \{t_n\})$ as

$$u \sim v \ (u, v \in V_{t_n}) \quad \Leftrightarrow \quad \begin{cases} u = v, & \text{if } n = 0, \\ M_u^{(t_n, t_{n-1}+1)} = M_v^{(t_n, t_{n-1}+1)}, & \text{if } n \in \mathbb{N}. \end{cases}$$

Using this equivalence relation, we construct W by

$$W_n = V_{t_n} / \sim .$$

For $x \in W_{n-1}$ and $w \in W_n$, define $N_{wx}^{(n)}$ as

$$N_{wx}^{(n)} = \sum_{v \in x} M_{uv}^{(t_n, t_{n-1}+1)}, \quad \text{where } u \in w.$$

(In the case of $n = 1$, we put $v_0 \in w_0$ where $W_0 = \{w_0\}$, $V_0 = \{v_0\}$.) Note that this definition is independent of the choice of $u \in w$.

Remark 4.1

(1) We give an example of (stationary) Bratteli diagrams satisfying the conditions above. For any $n \in \mathbb{N}$, set $t_n = n$, $V_n = \{1, 2, 3, 4, 5, 6\}$ and $W_n = \{w_1, w_2, w_3\}$. Incidence matrices $M^{(n)}$ and $N^{(n)}$ are defined by

$$M^{(1)} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}, N^{(1)} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, M^{(n)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 4 & 5 \end{bmatrix}, N^{(n)} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 8 & 10 \\ 3 & 8 & 10 \end{bmatrix} \ (n \geq 2).$$

Then we see that $1, 2 \in w_1$, $3, 5 \in w_2$ and $4, 6 \in w_3$.

(2) In this example, $w_2 \neq w_3$ but $N_{w_2}^{(n)} = N_{w_3}^{(n)}$.

Proposition 4.2 *Suppose $\mathcal{B} = (V, E, \{M^{(n)}\})$ is a simple Bratteli diagram and $\{t_n\}$ is a sequence of telescoping depth satisfying that all $M^{(t_n, t_{n-1}+1)}$'s are positive matrices. Suppose C is the diagram constructed above. Then the following statements hold:*

(1) *for any $n \in \mathbb{N}$ and $s \in \mathbb{N}$, $\#\{w \in W_n \mid |r^{-1}(w)| \leq s\} < 2^s$,*

(2) *for any $v \in w \in W$, $|\mathcal{P}(v)| = |\mathcal{P}(w)|$,*

(3) *for any $0 \leq r < 1$, there exists $K \in \mathbb{N}$ such that $\sum_{w \in W_n} r^{|\mathcal{P}(w)|} < 1$ for all $n \geq K$,*

(4) *$\mathcal{B} \sim C$.*

Proof. (1) Since

$$\{M_u^{(t_n, t_{n-1}+1)} \mid u \in V_{t_n}\} \subset$$
$$\left\{ (n_v) \in \mathbb{N}^{|V_{t_{n-1}}|} \;\middle|\; \sum_{v \in V_{t_{n-1}}} n_v = \sum_{v \in V_{t_{n-1}}} M_{uv}^{(t_n, t_{n-1}+1)},\ u \in V_{t_n} \right\}$$

and

$$\{|r^{-1}(w)| \mid w \in W_n\} = \left\{ \sum_{v \in V_{t_{n-1}}} M_{uv}^{(t_n, t_{n-1}+1)} \;\middle|\; u \in V_{t_n} \right\},$$

we have

$$\#\{w \in W_n \mid |r^{-1}(w)| = s\} \leq \#\left\{(n_v) \in \mathbb{N}^{|V_{t_{n-1}}|} \;\middle|\; \sum_{v \in V_{t_{n-1}}} n_v = s\right\} = \binom{s-1}{|V_{t_{n-1}}|-1}.$$

Then we have

$$\#\{w \in W_n \mid |r^{-1}(w)| \leq s\} \leq \sum_{i=|V_{t_{n-1}}|}^{s} \binom{i-1}{|V_{t_{n-1}}|-1} = \binom{s}{|V_{t_{n-1}}|} < 2^s,$$

where we used the formula $\binom{n-1}{r-1} = \binom{n}{r} - \binom{n-1}{r}$.

(2) In the case of $n = 1$, $|\mathcal{P}(v)| = M_{vv_0}^{(t_1,1)} = N_{ww_0}^{(1)} = |\mathcal{P}(w)|$ holds for any $v \in w \in W_1$. Suppose that for any $v \in x \in W_{n-1}$, $|\mathcal{P}(v)| = |\mathcal{P}(x)|$ holds. Then for $u \in w \in W_n$, we have

$$\begin{aligned}
|\mathcal{P}(u)| &= \sum_{v \in V_{t_{n-1}}} |\mathcal{P}(v)| M_{uv}^{(t_n,t_{n-1}+1)} = \sum_{x \in W_{n-1}} \left(\sum_{v \in x} |\mathcal{P}(v)| M_{uv}^{(t_n,t_{n-1}+1)} \right) \\
&= \sum_{x \in W_{n-1}} |\mathcal{P}(x)| \sum_{v \in x} M_{uv}^{(t_n,t_{n-1}+1)} = \sum_{x \in W_{n-1}} |\mathcal{P}(x)| N_{wx}^{(n)} = |\mathcal{P}(w)|.
\end{aligned}$$

(3) Put $p_{n-1}^{\min} = \min\{|P(x)| \mid x \in W_{n-1}\}$. By the simplicity of $\mathcal{B}$ and (2) above, it is easy to see that $p_{n-1}^{\min}$ is monotone increasing with respect to n. Using (1), we have

$$\sum_{w \in W_n} r^{|\mathcal{P}(w)|} \leq \sum_{w \in W_n} r^{p_{n-1}^{\min} |r^{-1}(w)|} < \sum_{s=1}^{\infty} r^{p_{n-1}^{\min} s} \times 2^s = \frac{2r^{p_{n-1}^{\min}}}{1 - 2r^{p_{n-1}^{\min}}} \to 0$$

as $n \to \infty$.

(4) We will construct a Bratteli diagram $\hat{\mathcal{B}} = (\hat{V}, \hat{E}, \{\hat{M}^{(n)}\})$ so that $\hat{\mathcal{B}}_{\text{even}}$ corresponds to $(\mathcal{B}, \{t_n\})$ and $\hat{\mathcal{B}}_{\text{odd}}$ corresponds to $\mathcal{C}$. For $n \in \mathbb{N}$, we put $\hat{V}_{2n-1} = W_n$, $\hat{V}_{2n} = V_{t_n}$ and define the incidence matrix $\hat{M}^{(n)}$ as

$$\hat{M}_{wv}^{(2n-1)} = M_{uv}^{(t_n,t_{n-1}+1)}, \quad \text{where } u \in w,$$
$$\hat{M}_{vw}^{(2n)} = \delta_{vw}^{(n)}, \quad \text{where } \delta_{vw}^{(n)} = 1 \text{ if } v \in w, \text{ and } \delta_{vw}^{(n)} = 0 \text{ if } v \notin w.$$

We will check that $\hat{M}^{(2n,2n-1)} = M^{(t_n,t_{n-1}+1)}$ and $\hat{M}^{(2n+1,2n)} = N^{(n+1)}$ ($\hat{M}^{(0)} = N^{(0)} = [1]$ for convenience).

$$\begin{aligned}
\hat{M}_{uv}^{(2n,2n-1)} &= \sum_{w \in W_n} \hat{M}_{uw}^{(2n)} \hat{M}_{wv}^{(2n-1)} = \sum_{w \in W_n} \delta_{uw}^{(n)} M_{tv}^{(t_n,t_{n-1}+1)} \quad (t \in w) \\
&= M_{uv}^{(t_n,t_{n-1}+1)} \quad (\because\; u, t \in w \Rightarrow M_{uv}^{(t_n,t_{n-1}+1)} = M_{tv}^{(t_n,t_{n-1}+1)}),
\end{aligned}$$

$$\hat{M}_{wx}^{(2n+1,2n)} = \sum_{v \in \bar{V}_{t_n}} \hat{M}_{wv}^{(2n+1)} \hat{M}_{vx}^{(2n)} = \sum_{v \in \bar{V}_{t_n}} M_{tv}^{(t_{n+1},t_n+1)} \delta_{vx}^{(n)} \quad (t \in w)$$
$$= \sum_{v \in x} M_{tv}^{(t_{n+1},t_n+1)} = N_{wx}^{(n+1)} \quad (\because\ t \in w).$$

Remark 4.3

(1) In the example of Remark 4.1, $\hat{M}^{(n)}$ is the following.

$$\hat{M}^{(1)} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \hat{M}^{(2n)} = \begin{bmatrix} 1&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}, \quad \hat{M}^{(2n+1)} = \begin{bmatrix} 1&1&1&1&1&1 \\ 1&2&3&4&5&6 \\ 2&1&4&5&4&5 \end{bmatrix} \quad (n \in \mathbb{N}).$$

(2) Suppose $\mathcal{B}$ and $\mathcal{C}$ are Bratteli diagrams satisfying Proposition 4.2. Then there is an onto map $\Phi : E' \to F$, where $E' = \cup_{n=1}^{\infty} E_{t_n,t_{n-1}+1}$ is the edge set of $(\mathcal{B}, \{t_n\})$, such that

- $\Phi(E_{t_n,t_{n-1}+1}) = F_n$,
- for any $e \in E'$, $s(e) \in s(\Phi(e))$ and $r(e) \in r(\Phi(e))$,
- for any $v \in w \in W_n$, Φ gives a bijection between $\{e \in E_{t_n,t_{n-1}+1} \mid r(e) = v\}$ and $r^{-1}(w)$,
- for any $\tilde{e} \in F_n$ and $e, e' \in \Phi^{-1}(\tilde{e})$, $s(e) = s(e')$.

Then we can define a map $\Phi^n : \mathcal{P}(V_{t_n}) \to \mathcal{P}(W_n)$ as

$$x_{[1,t_n]} \mapsto \Phi(x_{[1,t_1]})\Phi(x_{(t_1,t_2]}) \dots \Phi(x_{(t_{n-1},t_n]}).$$

We see that

(i) Φ^n is surjective,

(ii) the restricted map $\Phi^n|_{\mathcal{P}(v)}$ is a bijection between $\mathcal{P}(v)$ and $\mathcal{P}(w)$ where $v \in w$,

(iii) for any $p \in \mathcal{P}(W_n)$ and $x_{[1,t_n]}, x'_{[1,t_n]} \in (\Phi^n)^{-1}(p)$, $x_{[1,t_{n-1}]} = x'_{[1,t_{n-1}]}$ holds.

Using Φ^n's, we define $\varphi : X_{\mathcal{B}} \to X_{\mathcal{C}}$ as

$$\varphi((x_n)_{n \in \mathbb{N}}) = (y_n)_{n \in \mathbb{N}} \quad \Leftrightarrow \quad \text{for any } n \in \mathbb{N},\ \Phi^n(x_{[1,t_n]}) = y_{[1,n]}.$$

Then we can show that φ is bijective by the following. By (i) φ is surjective. For any fixed $y \in X_{\mathcal{C}}$, the number of paths in $\mathcal{P}(V_{t_n})$ corresponding to $y_{[1,n]}$ via Φ^n (i.e., $|(\Phi^n)^{-1}(y_{[1,n]})|$) is $\#\{v \in V_{t_n} \mid v \in r(y_n)\}$ because of (ii). However, by (iii) source vertices of each edge in E_{t_{n+1},t_n+1} corresponding to y_{n+1} via Φ are a same vertex. Therefore considering preimage of $y_{[1,n+1]}$ via Φ^{n+1}, we can choose uniquely the path in $\mathcal{P}(V_{t_n})$ corresponding to $y_{[1,n]}$ via Φ^n. This means φ is injective.

(3) φ preserves the cofinal relation. I.e,

$$x \neq x' \in X_{\mathcal{B}} \text{ and } \forall n \geq t_N, x_n = x'_n \implies \forall n \geq N,\ \varphi(x)_n = \varphi(x')_n.$$

Therefore, if we assign any proper order $\leq_{\mathcal{B}}$, $\leq_{\mathcal{C}}$ on $\mathcal{B}$, $\mathcal{C}$ respectively, φ is an orbit equivalence map. Moreover if $\leq_{\mathcal{B}}$ and $\leq_{\mathcal{C}}$ satisfies $\varphi(x_{\min}) = y_{\min}$ and $\varphi(x_{\max}) = y_{\max}$, φ is a strong orbit equivalence map.

(4) Suppose f is a simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_{t_{n-1}})$. Then $f \circ \varphi^{-1}$ is a simple function on $X_{\mathcal{C}}$ but not based on $\mathcal{P}(W_{n-1})$ in general. Indeed, $f \circ \varphi^{-1}$ is based on $\mathcal{P}(W_{n-1})$ if and only if $f[p]_{\mathcal{B}} = f[p']_{\mathcal{B}}$ for any $p, p' \in \mathcal{P}(V_{t_{n-1}})$ with $\Phi^n(p) = \Phi^n(p')$. However, $f \circ \varphi^{-1}$ is based on $\mathcal{P}(W_n)$. We regard f as a simple function based on $\mathcal{P}(V_{t_n})$ by

$$f(x) = f[p_{[1,t_{n-1}]}]_{\mathcal{B}} \quad \text{if } x \in [p]_{\mathcal{B}},\ p \in \mathcal{P}(V_{t_n}).$$

By the condition (iii), $\Phi^n(x_{[1,t_n]}) = \Phi^n(x'_{[1,t_n]})$ implies $x_{[1,t_{n-1}]} = x'_{[1,t_{n-1}]}$. Therefore

$$f \circ \varphi^{-1}(y) = f[p_{[1,t_{n-1}]}]_{\mathcal{B}} \quad \text{if } y \in [\Phi^n(p)]_{\mathcal{C}}$$

does not depend on a choice of $p \in P(V_{t_n})$ and is a simple function based on $\mathcal{P}(W_n)$.

Here we introduce the "converse" construction of the vertex amalgamation, which is called *the vertex splitting*.

The vertex splitting construction of $\tilde{\mathcal{B}}$. Suppose $\mathcal{C} = (W, F, \{N^{(n)}\})$ is a simple Bratteli diagram. Suppose $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\})$ satisfies

- $\tilde{V}_n = \cup_{w \in W_n} \tilde{V}_{n,w}$ as disjoint union and $\tilde{V}_{n,w} \neq \emptyset$, (I.e., we split w into $|\tilde{V}_{n,w}|$ vertices in $\tilde{V}_n$.)
- for any $u, v \in \tilde{V}_{n,w}$, $\tilde{M}_u^{(n)} = \tilde{M}_v^{(n)}$,
- for $w, x \in W_n$ with $w \neq x$, then $\tilde{M}_u^{(n)} \neq \tilde{M}_v^{(n)}$ for $u \in \tilde{V}_{n,w}$ and $v \in \tilde{V}_{n,x}$
- for any $u \in \tilde{V}_{n,w}$, $\sum_{v \in \tilde{V}_{n-1,x}} \tilde{M}_{uv}^{(n)} = N_{wx}^{(n)}$.

Remark 4.4 In the case of the vertex amalgamation construction, $\mathcal{C}$ is uniquely determined. However, in the case of the vertex splitting construction, there is an ambiguity of a number of vertices and hence $\tilde{\mathcal{B}}$ is not uniquely determined.

Proposition 4.5 $\tilde{\mathcal{B}} \sim \mathcal{C}$.

Proof. This follows Proposition 4.2 by putting $\tilde{\mathcal{B}} = \mathcal{B}$ and $\{t_n\} = \mathbb{Z}_+$.

Remark 4.6

(1) Suppose $\tilde{\mathcal{B}}$ and $\mathcal{C}$ are simple Bratteli diagrams satisfying the vertex splitting construction. By similar arguments of Remark 4.3 (2), we have a bijection $\tilde{\varphi} : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$ preserving the cofinal relation. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are simple Bratteli diagrams satisfying the vertex amalgamation construction and $\mathcal{B}$ and $\tilde{\mathcal{B}}$ have proper orders $\leq_{\mathcal{B}}$, and $\leq_{\tilde{\mathcal{B}}}$ respectively satisfying

$$\varphi(x_{\min}) = \tilde{\varphi}(\tilde{x}_{\min}) \text{ and } \varphi(x_{\max}) = \tilde{\varphi}(\tilde{x}_{\max}).$$

Then $\tilde{\varphi}^{-1} \circ \varphi$ is a strong orbit equivalence map between $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and $(X_{\tilde{\mathcal{B}}}, \lambda_{\tilde{\mathcal{B}}})$.

(2) Let $\tilde{\Phi}^n : \mathcal{P}(\tilde{V}_n) \to \mathcal{P}(W_n)$ be an onto map which induces a conjugacy $\tilde{\varphi}$ (see Remark 4.3 (2)) and h be a simple function on $X_{\mathcal{C}}$ based on $\mathcal{P}(W_n)$. Then we see that for any $\tilde{x}, \tilde{x}' \in X_{\tilde{\mathcal{B}}}$ with $\tilde{\Phi}^n(\tilde{x}_{[1,n]}) = \tilde{\Phi}^n(\tilde{x}'_{[1,n]}) = q$,

$$h \circ \tilde{\varphi}(\tilde{x}) = h \circ \tilde{\varphi}(\tilde{x}') = h[q]_{\mathcal{C}}.$$

This implies that for any $v, v' \in \tilde{V}_{n,w}$,

$$\sum_{p \in \mathcal{P}(v)} h \circ \tilde{\varphi}[p]_{\tilde{\mathcal{B}}} = \sum_{p \in \mathcal{P}(v')} h \circ \tilde{\varphi}[p]_{\tilde{\mathcal{B}}} = \sum_{q \in \mathcal{P}(w)} h[q]_{\mathcal{C}}.$$

5. Proof of Theorem 1.1

5.1. Requirements of a simple Bratteli diagram for (Y, ψ).

By Theorem 9.7 in [W1], for a topological dynamical system (X, T) and potential function $f \in C(X, \mathbb{R})$,

$$h(T) + \inf f \leq P(T, f) \leq h(T) + \sup f$$

and so $P(T, f) = \infty$ iff $h(T) = \infty$. In the case of $\alpha = \infty$, there exists a Cantor minimal system (Y, ψ) strongly orbit equivalent to (X, ϕ) such that $h(\psi) = \infty$ (see [S2]). This means

$$P(\psi, f \circ \theta^{-1}) = \infty.$$

So we only consider the case where α is finite. Let $\mathcal{B} = (V, E, \{M^{(n)}\}, \geq)$ be a properly ordered Bratteli diagram which is a representation of (X, ϕ). So we identify (X, ϕ) with $(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$. From the simplicity of diagram, we may assume that all $M^{(n)}$'s are positive matrices. We only consider within a strong orbit equivalence class of (X, ϕ). So applying Proposition 4.2 to $\mathcal{B}$, we may also assume that

$$\forall n, s \in \mathbb{N},\ \#\{v \in V_n \mid |r^{-1}(v)| \leq s\} \leq 2^s, \tag{5.1}$$

$$0 \leq \forall r < 1,\ \exists K \in \mathbb{N} \text{ s.t.} \forall n \geq K,\ \sum_{v \in V_n} r^{|\mathcal{P}(v)|} < 1. \tag{5.2}$$

Choose any decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ satisfying $0 < \frac{1}{3}\varepsilon_n < \varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and fix it. Now we will construct a properly ordered Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}, \tilde{\geq})$ which is a representation of (Y, ψ). First, applying the vertex amalgamation construction to $(\mathcal{B}, \{t_n\})$ where $\{t_n\}$ is some suitable telescoping depths, we have a based Bratteli diagram $\mathcal{C} = (W, F, \{N^{(n)}\})$ with $\mathcal{C} \sim \mathcal{B}$ (see Proposition 4.2). Second, applying the vertex splitting construction to $\mathcal{C}$, we have $\tilde{\mathcal{B}}$ with $\tilde{\mathcal{B}} \sim \mathcal{C}$ (see Proposition 4.5

5.2. Preliminary

In this subsection, we will introduce some lemmas.

Lemma 5.1 *Suppose that $N, A, Q \in \mathbb{N}$ with $A \geq 3$, $R \in \mathbb{Z}_+$ and $1 < r < 2$ satisfy the following conditions:*

(1) $N - 2 = (A-2)Q + R$ *and* $0 \leq R < A - 2$,

(2) $(r-1)Q > A$, $(2-r)Q \geq 1$ *and* $\frac{(r-1)(N-A)}{A-2} > 1$.

Then the following inequality holds.

$$\#\left\{(n_i) \in \mathbb{N}^{A-2} \mid \sum_{i=1}^{A-2} n_i = N-2,\ n_i < rQ\right\} \geq \left(\frac{(r-1)(N-2)}{A-2} - r\right)^{A-3}$$

Proof. Let $\{l_i\}_{i=1}^{A-3}$ be a set of non-negative integers with $l_i < (r-1)Q$. Define $\{n_i\}_{i=1}^{A-2} \subset \mathbb{N}$ as

$$n_i = \begin{cases} Q + R - l_1 & \text{if } i = 1, \\ Q + l_{i-1} - l_i & \text{if } 2 \leq i < A-2, \\ Q + l_{A-3} & \text{if } i = A-2. \end{cases}$$

Then we can easily verify that $\{n_i\}$ satisfies $\sum_{i=1}^{A-2} n_i = L - 2$ and by condition (2), $1 \leq n_i < rQ$ holds for each i. Moreover it is easy to check that the map $(l_1, l_2, \cdots, l_{A-1}) \mapsto (n_1, n_2, \cdots, n_{A-2})$ is injective. Let [ ] denote the Gauss symbol (i.e. $[x]$ is the integer part of x). So we get

$$\#\left\{(n_i) \in \mathbb{N}^{A-2} \mid \sum_{i=1}^{A-2} n_i = N-2,\ n_i < rQ\right\} \geq \#\left\{(l_i) \in \mathbb{Z}_+^{A-3} \mid l_i < (r-1)Q\right\}$$
$$\geq ([(r-1)Q])^{A-3} = \left(\left[\frac{(r-1)(N-2-R)}{A-2}\right]\right)^{A-3} \geq \left(\left[\frac{(r-1)(N-A)}{A-2}\right]\right)^{A-3}$$
$$\geq \left(\frac{(r-1)(N-A)}{A-2} - 1\right)^{A-3} = \left(\frac{(r-1)(N-2)}{A-2} - r\right)^{A-3}.$$

So we finish the proof.

We will use the following notations.

$$\bar{M}_v^{(n)} = \sum_{u \in V_{n-1}} M_{vu}^{(n)}, \quad \hat{M}_{vu}^{(n)} = \frac{M_{vu}^{(n)}}{\bar{M}_v^{(n)}}.$$

Lemma 5.2 *Suppose that $(V, E, \{M^{(n)}\})$ is a simple Bratteli diagram with positive matrices and $N \in \mathbb{N}$ is given. Then there exists $\{c_u\}_{u \in V_{N-1}}$ with $0 < c_u < 1$ such that*

$$c_u \leq \inf\{\hat{M}_{vu}^{(N+k,N)} \mid v \in V_{N+k},\ k \in \mathbb{N}\}. \tag{5.3}$$

Proof. For any $k \in \mathbb{N}$ and $v \in V_{N+k}$,

$$\bar{M}_v^{(N+k,N)} < \|M^{(N)}\| \times \bar{M}_v^{(N+k,N+1)} \text{ where } \|M^{(N)}\| \equiv \sum_{u,v} M_{vu}^{(N)}. \tag{5.4}$$

Also, the following inequality holds.

$$M_{vu}^{(N+k,N)} > \bar{M}_v^{(N+k,N+1)} \times \min_{l \in V_N} M_{lu}^{(N)} \tag{5.5}$$

From (5.4) and (5.5), we get $\hat{M}_{vu}^{(N+k,N)} > c_u$ where $c_u = (\min_{l \in V_N} M_{lu}^{(N)})/\|M^{(N)}\|$. It is clear that $0 < c_u < 1$ for all $u \in V_{N-1}$. Therefore (5.3) holds.

Lemma 5.3 *For all $n \in \mathbb{N}$, $\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+2}{e}\right)^{n+2}$.*

Proof. If $n = 1$, the inequality holds trivially. If $n \geq 2$, then $e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} > \frac{n^n}{n!}$. Therefore the left part of the inequality holds. Next, we can calculate

$$\log(n+1)! = \sum_{k=1}^{n+1} \log k < \sum_{k=1}^{n+1} \int_k^{k+1} \log x dx = (n+2)\log(n+2) - (n+1).$$

Since $\log(n+1) > 1$ for $n \geq 2$, we get

$$\begin{aligned} \log n! &< (n+2)\log(n+2) - (n+1) - \log(n+1) \\ &< (n+2)\{\log(n+2) - 1\} = \log\left((n+2)/e\right)^{n+2}. \end{aligned}$$

So the right part of the inequality also holds.

Let f be a function of $X_{\mathcal{B}}$. For $x \in X_{\mathcal{B}}$ and $m \in \mathbb{N}$, put

$$S(f, x, m) = \frac{1}{m} \sum_{i=0}^{m-1} f(\lambda_{\mathcal{B}}{}^i x).$$

Lemma 5.4 *Suppose $\mathcal{B} = (V, E, \geq)$ is a properly ordered Bratteli diagram, f is a simple function on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_N)$. For any $\beta > \exp(\sup\{\int f d\mu \mid \mu \in \mathcal{M}(X_{\mathcal{B}}, \lambda_{\mathcal{B}})\})$, there exists $N' \geq N$ such that for any $n \geq N'$ and $v \in V_n$,*

$$\beta^{|\mathcal{P}(v)|} > \exp\left(\sum_{p \in \mathcal{P}(v)} f[p_{[1,N]}]_{\mathcal{B}}\right).$$

Proof. Suppose this lemma is false. Then there are infinitely many n's and $v_n \in V_n$ so that

$$\beta^{|\mathcal{P}(v_n)|} \leq \exp\left(\sum_{p\in\mathcal{P}(v_n)} f[\tau_N p]_{\mathcal{B}}\right) \Leftrightarrow \exp\left(\log\beta - S(f, x_n, |\mathcal{P}(v_n)|)\right) \leq 1,$$

where $x_n \in X_{\mathcal{B}}$ is in the minimal path of $\mathcal{P}(v_n)$. Define $\mu_n \in \mathcal{M}(X_{\mathcal{B}})$ as

$$\mu_n = \frac{1}{|\mathcal{P}(v_n)|}\sum_{i=0}^{|\mathcal{P}(v_n)|-1} \delta_{\lambda_{\mathcal{B}}{}^i x_n}.$$

Choose subsequence $\{n_i\}$ so that $\{S(f, x_{n_i}, |\mathcal{P}(v_{n_i})|)\}$ is convergent and $\{\mu_{n_i}\}$ is convergent in the weak* topology on $\mathcal{M}(X_{\mathcal{B}})$. Let $\mu = \lim_{i\to\infty}\mu_{n_i}$. By Theorem 6.9 in [W1], we see that $\mu \in \mathcal{M}(X_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and

$$S(f, x_{n_i}, |\mathcal{P}(v_{n_i})|) = \int f d\mu_{n_i} \to \int f d\mu \quad (i \to \infty).$$

Therefore we have

$$\exp\left(\log\beta - \int f d\mu\right) \leq 1.$$

This contradicts $\beta > \exp\left(\sup\left\{\int f d\mu \mid \mu \in \mathcal{M}(X_{\mathcal{B}}, \lambda_{\mathcal{B}})\right\}\right)$.

5.3. The construction of a based diagram $\mathcal{C}$.

If $\{t_n\}$ is decided, we can construct C by the vertex amalgamation construction. Then, we define $\varphi : X_{\mathcal{B}} \to X_{\mathcal{C}}$ as Remark 4.3 (2) and a simple function f_n on $X_{\mathcal{B}}$ based on $\mathcal{P}(V_{t_n})$ as

$$f_n(x) = \min\{f(y) \mid y \in [p]_{\mathcal{B}}\} \quad \text{where } x \in [p]_{\mathcal{B}} \text{ and } p \in \mathcal{P}(V_{t_n}).$$

(Set $\mathcal{P}(V_0) = \emptyset$ and $[\emptyset]_{\mathcal{B}} = X_{\mathcal{B}}$. Then $f_0(x) = \min\{f(y) \mid y \in X_{\mathcal{B}}\}$.) We see that

- $\{f_n\}$ is monotone increasing and $\lim_{n\to\infty}\|f - f_n\| = 0$,
- $f_{n-1}\circ\varphi^{-1}$ is a simple function on $X_{\mathcal{C}}$ based on $\mathcal{P}(W_n)$ (Remark 4.3 (4)),
- for any $v \in w \in W_n$,

$$\sum_{p\in\mathcal{P}(v)} f_{n-1}[p_{[1,t_{n-1}]}]_{\mathcal{B}} = \sum_{q\in\mathcal{P}(w)} f_{n-1}\circ\varphi^{-1}[q]_{\mathcal{C}} \tag{5.6}$$

  (See Remark 4.3 (4). Put $p_{[1,0]} = \emptyset$.).

Define $\Gamma_n[w]$ as

$$\Gamma_n[w] = \exp\left(\sum_{p\in\mathcal{P}(v)} f_{n-1}[p_{[1,t_{n-1}]}]_{\mathcal{B}}\right), \tag{5.7}$$

where $v \in w \in W_n$. Now, we will decide $\{t_n\}$ by induction.

The 1st step. Put $t_0 = 0$. Applying Lemma 5.4 to f_0 and $\mathcal{B}$, there exists $t_1 \in \mathbb{N}$ satisfying

$$\left(\alpha + \frac{1}{3}\varepsilon_1\right)^{|\mathcal{P}(v)|} > \exp\left(\sum_{p\in\mathcal{P}(v)} f_0[p_{[1,0]}]_{\mathcal{B}}\right) = \exp\left(|\mathcal{P}(v)| \times \min\{f(y) \mid y \in X_{\mathcal{B}}\}\right),$$
$$\left(\frac{\alpha+\varepsilon_2}{\alpha+\frac{1}{3}\varepsilon_2}\right)^{|\mathcal{P}(v)|} > 2$$

for all $v \in V_{t_1}$. (The second part of inequality above holds because $\min_{v\in V_{t_1}} |\mathcal{P}(v)|$ is monotone increasing with respect to t_1.) We fix t_1. Then we can construct W_1 and $N^{(1)}$ of C by the vertex amalgamation construction. Since $|\mathcal{P}(w)| = |\mathcal{P}(v)|$ holds for $v \in w$, the first part of inequality above is equivalent to $\left(\alpha + \frac{1}{3}\varepsilon_1\right)^{|\mathcal{P}(w)|} > \Gamma_1[w]$ for any $w \in W_1$. Let $\{A_w^{(1)} \in \mathbb{N} \mid w \in W_1\}$ satisfy

$$A_w^{(1)} > 2 + \max\left\{\frac{(\alpha+\varepsilon_1)^{|\mathcal{P}(w)|}}{\Gamma_1[w]}, |V_{t_1,w}|\right\},$$

where $V_{t_1,w} = \{v \in V_{t_1} \mid v \in w\}$. Then there exists a unique number $\alpha_1 > \alpha + \varepsilon_1$ such that

$$\sum_{w\in W_1} \frac{A_w^{(1)}\Gamma_1[w]}{(\alpha_1)^{|\mathcal{P}(w)|}} = 1.$$

Choose any $\varepsilon_0 > \alpha_1 - \alpha$ and fix it.

The n-th step. For $n \geq 2$, suppose the $(n-1)$-th step data are given by the following: For any $w \in W_{n-1}$,

(D_{n-1}-1) $\left(\frac{\alpha+\varepsilon_n}{\alpha+\varepsilon_n/3}\right)^{|\mathcal{P}(w)|} > 2$,

(D_{n-1}-2) $(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|} < (A_w^{(n-1)} - 2)\Gamma_{n-1}[w]$,

(D_{n-1}-3) $|V_{t_{n-1},w}| < A_w^{(n-1)} - 2$.

Choose $r_w \in \mathbb{R}$ satisfying (5.8) and fix it.

$$1 < r_w < \min\left(\frac{3}{2}, \frac{(A_w^{(n-1)} - 2)\Gamma_{n-1}[w]}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(w)|}}\right) \tag{5.8}$$

For any fixed $t_n > t_{n-1}$, we can temporarily construct W_n and $N^{(n)}$ by the vertex amalgamation construction. Define $Q_{xw} \in \mathbb{N}$ and $R_{xw} \in \mathbb{Z}_+$ to be the unique numbers such that

$$N_{xw}^{(n)} - 2 = (A_w^{(n-1)} - 2)Q_{xw} + R_{xw} \quad \text{and} \quad 0 \leq R_{xw} < A_w^{(n-1)} - 2. \tag{5.9}$$

Define B_x, C_{xw} and D_{xw} as

$$B_x = \frac{((\bar{N}_x^{(n)} - 2)/e)^{\bar{N}_x^{(n)}-2}}{\prod_{w\in W_{n-1}}((r_w Q_{xw} + 2)/e)^{N_{xw}^{(n)}+2A_w^{(n-1)}}}, \quad \left(\bar{N}_x^{(n)} := \sum_{w\in W_{n-1}} N_{xw}^{(n)}\right)$$

$$C_{xw} = \left\{ (n_v) \in \mathbb{N}^{|V_{t_{n-1},w}|} \;\middle|\; \sum_{v\in w} n_v = N_{xw}^{(n)} \right\},$$

$$D_{xw} = \left\{ (n_i) \in \mathbb{N}^{A_w^{(n-1)}-2} \;\middle|\; \sum_{i=1}^{A_w^{(n-1)}-2} n_i = N_{xw}^{(n)} - 2,\ 1 \le n_i < r_w Q_{xw} \right\}.$$

Now we will show that Claim 5.5 holds for sufficiently large t_n.

Claim 5.5 *For any $x \in W_n$,*

(1) $\Gamma_n[x] < (\alpha + \frac{1}{3}\varepsilon_n)^{|\mathcal{P}(x)|}$,

(2) $B_x\Gamma_n[x](\alpha + \varepsilon_{n-1})^{-|\mathcal{P}(x)|} > 1$,

(3) *for any $w \in W_{n-1}$, $|C_{xw}| < |D_{xw}|$,*

(4) $|V_{t_n,x}|\Gamma_n[x] < (\alpha + \varepsilon_n)^{|\mathcal{P}(x)|}$,

(5) $\sum_{x\in W_n} \frac{2(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|}}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(x)|}} < 1$,

(6) $(\frac{\alpha+\varepsilon_{n+1}}{\alpha+\varepsilon_{n+1}/3})^{|\mathcal{P}(x)|} > 2$.

(1) f_{n-1} is a simple function based on $\mathcal{P}(V_{t_{n-1}})$. So applying Lemma 5.4 to f_{n-1} and $\mathcal{B}$, there exists $T > t_{n-1}$ such that for any $t_n > T$ satisfying

$$\left(\alpha + \frac{1}{3}\varepsilon_n\right)^{|\mathcal{P}(v)|} > \exp\left(\sum_{p\in\mathcal{P}(v)} f_{n-1}[p_{[1,t_{n-1}]}]_{\mathcal{B}}\right)$$

for all $v \in V_{t_n}$. Therefore by (5.7) and $|\mathcal{P}(v)| = |\mathcal{P}(x)|$ where $v \in x$, we have

$$\Gamma_n[x] < \left(\alpha + \frac{1}{3}\varepsilon_n\right)^{|\mathcal{P}(x)|}.$$

(2) Since

$$\Gamma_n[x] = \prod_{w\in W_{n-1}} \prod_{u\in w} \exp\left(\sum_{q\in\mathcal{P}(u)} f_{n-1}[q]_{\mathcal{B}} M_{vu}^{(t_n,t_{n-1}+1)}\right) \quad (v \in x)$$

$$\ge \prod_{w\in W_{n-1}} \exp\left(\sum_{q\in\mathcal{P}(u)} f_{n-2}[q_{[1,t_{n-2}]}]_{\mathcal{B}} N_{xw}^{(n)}\right) \quad (u \in w)$$

$$= \prod_{w \in W_{n-1}} (\Gamma_{n-1}[w])^{N^{(n)}_{xw}},$$

we have

$$\frac{B_x \Gamma_n[x]}{(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(x)|}} \geq E_x \times \prod_{w \in W_{n-1}} \left(\frac{(\bar{N}^{(n)}_x - 2)\Gamma_{n-1}[w]}{(r_w Q_{xw} + 2)(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} \right)^{N^{(n)}_{xw}},$$

where $E_x = \left\{ ((\bar{N}^{(n)}_x - 2)/e)^2 \prod_{w \in W_{n-1}} ((r_w Q_{xw} + 2)/e)^{2A^{(n-1)}_w} \right\}^{-1}$. Choose a small number $\varepsilon > 0$ satisfying (5.10) and fix it (See (5.8)).

$$\frac{(A^{(n-1)}_w - 2)\Gamma_{n-1}[w]}{r_w (\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} - \varepsilon > 1. \tag{5.10}$$

Since $\min_{x \in W_n} Q_{xw} \to \infty$ as $t_n \to \infty$, by (5.9) for a sufficiently large t_n,

$$\frac{(\bar{N}^{(n)}_x - 2)\Gamma_{n-1}[w]}{(r_w Q_{xw} + 2)(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} > \frac{(A^{(n-1)}_w - 2)\Gamma_{n-1}[w]}{r_w (\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} - \varepsilon. \tag{5.11}$$

From Lemma 5.2, there exists $\{c_u \mid u \in V_{t_{n-1}}\}$ with $0 < c_u < 1$ such that $\hat{M}^{(t_n, t_{n-1}+1)}_{vu} \geq c_u$ for any $v \in V_{t_n}$. Put $\hat{N}^{(n)}_{xw} = N^{(n)}_{xw} / \bar{N}^{(n)}_x$. For $v \in x$, we have

$$\begin{aligned} \hat{N}^{(n)}_{xw} &= \frac{\sum_{u \in w} M^{(t_n, t_{n-1}+1)}_{vu}}{\sum_{y \in W_{n-1}} \sum_{s \in y} M^{(t_n, t_{n-1}+1)}_{vs}} = \frac{\sum_{u \in w} M^{(t_n, t_{n-1}+1)}_{vu}}{\bar{M}^{(t_n, t_{n-1}+1)}_v} \\ &= \sum_{u \in w} \hat{M}^{(t_n, t_{n-1}+1)}_{vu} \geq \sum_{u \in w} c_u \equiv c'_w. \end{aligned} \tag{5.12}$$

Then $0 < c'_w < 1$ and (5.12) is independent of t_n and $x \in W_n$. As $\lim_{n \to \infty} n^{1/n} = 1$, it follows that $(E_x)^{1/\bar{N}^{(n)}_x} \to 1$ as $t \to \infty$. Therefore by (5.10), (5.11) and (5.12), the following inequality holds for sufficiently large t_n:

$$(E_x)^{1/\bar{N}^{(n)}} > \prod_{w \in W_{n-1}} \left(\frac{(A^{(n-1)}_w - 2)\Gamma_{n-1}[w]}{r_w (\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} - \varepsilon \right)^{-c'_w}.$$

Finally we get

$$\left(\frac{B_x \Gamma_n[x]}{(\alpha + \varepsilon_{n-1})^{|\mathcal{P}(x)|}} \right)^{1/\bar{N}^{(n)}_x} > \prod_{w \in W_{n-1}} \left(\frac{(A^{(n-1)}_w - 2)\Gamma_{n-1}[w]}{r_w (\alpha + \varepsilon_{n-1})^{|\mathcal{P}(w)|}} - \varepsilon \right)^{\hat{N}^{(n)}_{xw} - c'_w} \geq 1.$$

(3) It is easily seen that

$$|C_{xw}| = \binom{N^{(n)}_{xw} - 1}{|V_{t_{n-1}, w}| - 1} < (N^{(n)}_{xw})^{|V_{t_{n-1}, w}| - 1}. \tag{5.13}$$

Since $A_w^{(n-1)}$ and r_w are constant, we have $(r_w - 1)Q_{xw} > A_w^{(n-1)}$, $(2 - r_w)Q_{xw} \geq 1$ and $\frac{(r_w-1)(N_{xw}^{(n)} - A_w^{(n-1)})}{A_w^{(n-1)}-2} > 1$ for sufficiently large t_n. Therefore by Lemma 5.1, we see that

$$|D_{xw}| \geq \left(\frac{(r_w - 1)(N_{xw}^{(n)} - 2)}{A_w^{(n-1)} - 2} - r_w \right)^{A_w^{(n-1)}-3} . \tag{5.14}$$

Using (D_{n-1}-3), we obtain the following inequality for sufficiently large t_n:

$$(N_{xw}^{(n)})^{|V_{t_{n-1},w}|-1} < \left(\frac{(r_w - 1)(N_{xw}^{(n)} - 2)}{A_w^{(n-1)} - 2} - r_w \right)^{A_w^{(n-1)}-3} . \tag{5.15}$$

By (5.13), (5.14) and (5.15), we get the inequality of (3).

(4) As $|r^{-1}(v)| = \bar{M}_v^{(t_n)} \leq \bar{M}_v^{(t_n,t_{n-1}+1)} = |r^{-1}(x)|$ holds for $v \in V_{t_n,x}$, by using (5.1) we have

$$|V_{t_n,x}| \leq \#\{v \in V_{t_n} \mid |r^{-1}(v)| \leq |r^{-1}(x)|\} \leq 2^{|r^{-1}(x)|}.$$

By (D_{n-1}-1) and Claim 5.5 (1), for any $t_n > T$ (where T is defined in the proof of Claim 5.5 (1)) we have

$$\begin{aligned}(\alpha + \varepsilon_n)^{|\mathcal{P}(x)|} &> \Gamma_n[x] \prod_{w \in W_{n-1}} \left(\frac{\alpha + \varepsilon_n}{\alpha + \frac{1}{3}\varepsilon_n} \right)^{|\mathcal{P}(w)| N_{xw}^{(n)}} \\ &> \Gamma_n[x] \prod_{w \in W_{n-1}} 2^{N_{xw}^{(n)}} = \Gamma_n[x] 2^{|r^{-1}(x)|}.\end{aligned}$$

Therefore we get $|V_{t_n,x}|\Gamma_n[x] < (\alpha + \varepsilon_n)^{|\mathcal{P}(x)|}$.

(5) Choose any number η with $0 < \eta < \frac{1}{3}\varepsilon_{n-1}$ and fix it. Put $r = \frac{\alpha+\varepsilon_n+\eta}{\alpha+\varepsilon_{n-1}}$. Then $0 < r < 1$. By (5.2) for a sufficiently large t_n,

$$\sum_{v \in V_{t_n}} r^{|\mathcal{P}(v)|} < 1.$$

Clearly $|W_n| \leq |V_{t_n}|$. By Proposition 4.2 (3), we have

$$\sum_{x \in W_n} r^{|\mathcal{P}(x)|} < 1.$$

Since $|\mathcal{P}(x)| \to \infty$ holds as $t_n \to \infty$, for sufficiently large t_n, $2(\alpha + \varepsilon_n)^{|\mathcal{P}(x)|} < (\alpha + \varepsilon_n + \eta)^{|\mathcal{P}(x)|}$ holds. Therefore we have the inequality of (5).

(6) Since ε_{n+1} is independent with respect to t_n and $|\mathcal{P}(x)| \to \infty$ holds as $t_n \to \infty$, $(\frac{\alpha+\varepsilon_{n+1}}{\alpha+\varepsilon_{n+1}/3})^{|\mathcal{P}(x)|} > 2$ follows. □

Put t_n satisfying Claim 5.5. Then we can define $A_x^{(n)} \in \mathbb{N}$ as

$$(A_x^{(n)}-3)\Gamma_n[x] \le (\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} < (A_x^{(n)}-2)\Gamma_n[x] < A_x^{(n)}\Gamma_n[x] < 2(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} \tag{5.16}$$

because of Claim 5.5 (1). So we have the n-th step data by the following: For any $x \in W_n$,

(D_n-1) $(\frac{\alpha+\varepsilon_{n+1}}{\alpha+\varepsilon_{n+1}/3})^{|\mathcal{P}(x)|} > 2$,

(D_n-2) $(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|} < (A_x^{(n)}-2)\Gamma_n[x]$,

(D_n-3) $|V_{t_n,x}| < A_x^{(n)} - 2$.

5.4. The construction of $\tilde{\mathcal{B}}$.

In this subsection we will construct $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{(n)}\}, \tilde{\ge})$ satisfying Property 1.5 and for each $n \in \mathbb{N}$,

$$\alpha+\varepsilon_n < \alpha_n < \alpha+\varepsilon_{n-1} \quad \text{and} \quad \sum_{x\in W_n} \frac{|\tilde{V}_{n,x}|\Gamma_n[x]}{(\alpha_n)^{|\mathcal{P}(x)|}} = 1. \tag{5.17}$$

<u>The construction of $\tilde{V}_n$.</u> For $x \in W_n$, we set

$$|\tilde{V}_{n,x}| = A_x^{(n)}. \tag{5.18}$$

By the condition (D_n-3), $|\tilde{V}_{n,w}| \ge 3$ holds. Let $* \in W_n$ ($** \in W_n$ resp.) denote the vertex satisfying that the minimal path $x_{\min} \in X_{\mathcal{B}}$ (the maximal path $x_{\max} \in X_{\mathcal{B}}$ resp.) goes through some vertex in $V_{t_n,*}$ ($V_{t_n,**}$ resp.). We can choose any distinct vertices $v_{\min}^n \in \tilde{V}_{n,*}$ and $v_{\max}^n \in \tilde{V}_{n,**}$ (because of $|\tilde{V}_{n,w}| \ge 3$) and fix them. Then Property 1.5 (3) holds.

<u>The construction of $\tilde{M}^{(n)}$.</u> In the case of $n = 1$, for $v \in \tilde{V}_{1,w}$ we define

$$\tilde{M}_{vv_0}^{(1)} = N_{ww_0}^{(1)}$$

where $\tilde{V}_0 = \{v_0\}$ and $W_0 = \{w_0\}$. In the case of $n \ge 2$, we consider the following conditions with respect to $\tilde{M}^{(n)}$:

(c.0) If $x, x' \in W_n$ with $x \ne x'$, then $\tilde{M}_v^{(n)} \ne \tilde{M}_{v'}^{(n)}$, where $v \in \tilde{V}_{n,x}$ and $v' \in \tilde{V}_{n,x'}$.

(c.1) For any $v, v' \in \tilde{V}_{n,x}$, $\tilde{M}_v^{(n)} = \tilde{M}_{v'}^{(n)}$.

(c.2) For any $v \in \tilde{V}_{n,x}$,

$$(\tilde{M}_{vu}^{(n)})_{u\in\tilde{V}_{n-1,w}} \in \tilde{D}_{xw}$$

where $\tilde{D}_{xw}$ is defined by

$$\tilde{D}_{xw} = \left\{ (n_u) \in \mathbb{N}^{|\tilde{V}_w|} \mid \sum_{u\in\tilde{V}_w} n_u = N_{xw}^{(n)} - \delta_*(w) - \delta_{**}(w),\ n_u < r_w Q_{xw} \right\}$$

and

$$\delta_*(w) = \begin{cases} 1 & \text{if } w = *, \\ 0 & \text{if } w \neq *, \end{cases} \qquad \delta_{**}(w) = \begin{cases} 1 & \text{if } w = **, \\ 0 & \text{if } w \neq **. \end{cases}$$

(c.3) $\tilde{M}^{(n)}_{vv^{n-1}_{\min}} = \tilde{M}^{(n)}_{vv^{n-1}_{\max}} = 1$ for any $v \in \tilde{V}_n$.

It is easy to construct $\tilde{M}^{(n)}$ satisfying the conditions (c.1), (c.2) and (c.3) and these conditions imply that $\tilde{B}$ satisfies the assumptions of the vertex splitting construction and Property 1.5 (1), (4). Now we will show that we can construct it satisfying also the condition (c.0).

Suppose that $\tilde{M}^{(n)}$ satisfies only the conditions (c.1), (c.2) and (c.3). It is clear that if $N^{(n)}_w \neq N^{(n)}_x$, then $\tilde{M}^{(n)}_u \neq \tilde{M}^{(n)}_v$ where $u \in \tilde{V}_{n,w}$ and $v \in \tilde{V}_{n,x}$. In general, $x \neq x' \in W_n$ does not imply $N^{(n)}_x \neq N^{(n)}_{x'}$ (see Remark 4.1 (2)) and so we will show that for any $x \neq x' \in W_n$ with $N^{(n)}_x = N^{(n)}_{x'}$, we can reconstruct $\tilde{M}^{(n)}_v$ and $\tilde{M}^{(n)}_{v'}$ satisfying $\tilde{M}^{(n)}_v \neq \tilde{M}^{(n)}_{v'}$ for $v \in \tilde{V}_{n,x}$ and $v' \in \tilde{V}_{n,x'}$. By the construction of $N^{(n)}$, we see that

$$\#\left\{s \in W_n \mid N^{(n)}_s = N^{(n)}_x\right\} \leq \prod_{w \in W_{n-1}} |C_{xw}|. \tag{5.19}$$

As $\tilde{M}^{(n)}_v$ and $\tilde{M}^{(n)}_{v'}$ satisfy the condition (c.2), by Claim 5.5 (3) and (5.19) we have

$$\#\left\{s \in W_n \mid N^{(n)}_s = N^{(n)}_x\right\} \leq \prod_{w \in W_{n-1}} |D_{xw}| \leq \prod_{w \in W_{n-1}} |\tilde{D}_{xw}|. \tag{5.20}$$

The right part of the inequality (5.20) means what the maximum possible value for incidence vectors in $\mathbb{N}^{|\tilde{V}_{n-1}|}$ satisfying the condition (c.2) is. Therefore, we can choose incidence vectors satisfying $\tilde{M}^{(n)}_v \neq \tilde{M}^{(n)}_{v'}$.

<u>**The construction of $\tilde{\geq}$.**</u> We will check that we can construct $\tilde{\geq}$ on $\tilde{E}$ with the property that each $\tilde{E}_n$ has the minimal/maximal vertex property (Property 1.5 (2)) and each $\tilde{V}_n$ has distinct order lists (Property 1.5 (5)). For $x \in W_n$, define $\mathrm{Dist}(x) \in \mathbb{N}$ as

$$\mathrm{Dist}(x) = \frac{\left(\sum_{u \in \tilde{V}^*_{n-1}} \tilde{M}^{(n)}_{vu}\right)!}{\prod_{u \in \tilde{V}^*_{n-1}} \tilde{M}^{(n)}_{vu}!} = \frac{(\tilde{N}^{(n)}_x - 2)!}{\prod_{u \in \tilde{V}^*_{n-1}} \tilde{M}^{(n)}_{vu}!},$$

where $v \in \tilde{V}_{n,x}$ and $\tilde{V}^*_{n-1} = \tilde{V}_{n-1}/\{v^{n-1}_{\min}, v^{n-1}_{\max}\}$. $\mathrm{Dist}(x)$ means the maximal possible number of order lists of $v \in \tilde{V}_{n,x}$ satisfying Property 1.5 (2). Suppose $w \neq x$, $u \in \tilde{V}_{n,w}$ and $v \in \tilde{V}_{n,x}$. By the condition (c.0), if we assign any order on $r^{-1}(u)$, $r^{-1}(v)$ respectively, $\mathrm{List}(u) \neq \mathrm{List}(v)$ always holds. Therefore $\tilde{V}_n$ can have distinct order lists if and only if

$$\mathrm{Dist}(x) \geq |\tilde{V}_{n,x}|$$

for any x and hence we check this inequality. Since $\tilde{M}^{(n)}$ satisfies the conditions (c.2) and (c.3), using Claim 5.5 (2) and Lemma 5.3, we have

$$\begin{aligned}\frac{(\mathrm{Dist}(x)-3)\Gamma_n[x]}{(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|}} &> \frac{\left((\tilde{N}_x^{(n)}-2)/e\right)^{\tilde{N}_x^{(n)}-2}\Gamma_n[x]}{\prod_{u\in\tilde{V}_{n-1}^{*}}\left((\tilde{M}_{vu}^{(n)}+2)/e\right)^{\tilde{M}_{vu}^{(n)}+2}}\times(\alpha+\varepsilon_n)^{-|\mathcal{P}(x)|}\\ &> \frac{\left((\tilde{N}_x^{(n)}-2)/e\right)^{\tilde{N}_x^{(n)}-2}\Gamma_n[x]}{\prod_{w\in W_{n-1}}((r_wQ_{xw}+2)/e)^{N_{xw}^{(n)}+2|\tilde{V}_{n-1,w}|}}\times(\alpha+\varepsilon_{n-1})^{-|\mathcal{P}(x)|}\\ &= B_x\Gamma_n[x]>1,\end{aligned}$$

where $v\in\tilde{V}_{n,x}$. (We use the fact that if $n\geq 4$, then $n!-3>(\frac{n}{e})^n$ holds.) Therefore

$$\mathrm{Dist}(x)>(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|}\Gamma_n[x]^{-1}+3\geq|\tilde{V}_{n,x}|$$

because of (5.16) and (5.18).

<u>The check of (5.17).</u> By (5.16), (5.18) and Claim 5.5 (5), we have

$$\sum_{x\in W_n}\frac{|\tilde{V}_{n,x}|\Gamma_n[x]}{(\alpha+\varepsilon_{n-1})^{|\mathcal{P}(x)|}}<1.$$

The n-th step data (D_n-2) implies that $(\alpha+\varepsilon_n)^{|\mathcal{P}(x)|}<|\tilde{V}_{n,x}|\Gamma_n[x]$. Therefore there exists unique α_n with $\alpha+\varepsilon_n<\alpha_n<\alpha+\varepsilon_{n-1}$ such that

$$\sum_{x\in W_n}\frac{|\tilde{V}_{n,x}|\Gamma_n[x]}{(\alpha_n)^{|\mathcal{P}(x)|}}=1.$$

5.5. The check of properties for (Y,ψ)

Finally we will show that the Cantor minimal system $(X_{\tilde{\mathcal{B}}},\lambda_{\tilde{\mathcal{B}}})$ for (Y,ψ) satisfies the conditions of Theorem 1.1. Define $\tilde{\varphi}:X_{\tilde{\mathcal{B}}}\to X_{\mathcal{C}}$ as Remark 4.6 (1). Then $\theta=\tilde{\varphi}^{-1}\circ\varphi:X_{\mathcal{B}}\to X_{\tilde{\mathcal{B}}}$ is a strong orbit equivalence map between (X,ϕ) and (Y,ψ). Define functions g_n and g on $X_{\tilde{\mathcal{B}}}$ as

$$g_n=f_{n-1}\circ\varphi^{-1}\circ\tilde{\varphi}=f_{n-1}\circ\theta^{-1}\quad\text{and}\quad g=f\circ\varphi^{-1}\circ\tilde{\varphi}=f\circ\theta^{-1}.$$

Then g_n is a simple function on $X_{\tilde{\mathcal{B}}}$ based on $\mathcal{P}(\tilde{V}_n)$ and $\lim_{n\to\infty}\|g_n-g\|=0$. Moreover, since $f_{n-1}\circ\varphi^{-1}$ is a simple function on $X_{\mathcal{C}}$ based on $\mathcal{P}(W_n)$, for any $w\in W_n$ and $v,v'\in\tilde{V}_{n,w}$ we have

$$\sum_{p\in\mathcal{P}(v)}g_n[p]_{\tilde{\mathcal{B}}}=\sum_{p\in\mathcal{P}(v')}g_n[p]_{\tilde{\mathcal{B}}}=\sum_{q\in\mathcal{P}(w)}f_{n-1}\circ\varphi^{-1}[q]_{\mathcal{C}}\tag{5.21}$$

(see Remark 4.6 (2)). So we define $\Gamma_n(v)$ $(v \in \bar{V}_n)$ as

$$\Gamma_n(v) = \exp\left(\sum_{p\in\mathcal{P}(v)} g_n[p]_{\bar{\mathcal{B}}}\right),$$

then for $v \in \bar{V}_{n,w}$, $\Gamma_n(v) = \Gamma_n[w]$ because of (5.21). Therefore $\bar{\mathcal{B}}$ satisfies the following conditions: For each $n \in \mathbb{N}$,

(1) $\alpha+\varepsilon_n < \alpha_n < \alpha+\varepsilon_{n-1}$ and $\displaystyle\sum_{w\in W_n} \frac{|\bar{V}_{n,w}|\Gamma_n[w]}{(\alpha_n)^{|\mathcal{P}(w)|}} = 1 \left(\Leftrightarrow \sum_{v\in\bar{V}_n} \frac{\Gamma_n(v)}{(\alpha_n)^{|\mathcal{P}(v)|}} = 1\right)$,

(2) $\bar{\mathcal{B}}$ satisfies Property 1.5.

Applying Theorem 3.8 to $\bar{\mathcal{B}}$, we have

$$P(\psi, f\circ\theta^{-1}) = P(\lambda_{\bar{\mathcal{B}}}, g) = \lim_{n\to\infty} \log\alpha_n = \log\alpha.$$

Finally by Theorem 2.4, (Y,ψ) is topologically conjugate to a subshift.

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