Buchberger stratification on the space of polynomials

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Abstract. In this paper, we consider the stratification on the space of polynomials, with a fixed set of monomials, induced from the notion of G-types, which encodes the process of Buchberger algorithm for obtaining Gröbner bases. We show that this indeed gives a finite stratification by locally closed subspaces, from which we deduce several finiteness results on Gröbner bases.

1 Introduction

The notion of Gröbner bases and the algorithm to obtain them are extremely useful, both in theories and for computational purposes, and have been a subject of extensive studies in various fields of mathematics. One of the interesting aspects of Gröbner bases is the several finiteness and boundedness properties, e.g., upper-bound of the number of polynomials in a Gröbner base. The existence of such an upper-bound is already known; actually an upper-bound itself for the maximal degree has been obtained in recent works such as [2][4]. But our point of the interests lies rather in, whatever the upper-bound is, the geometric structures behind these finiteness and boundedness properties. This naturally leads us to study the space of ideals (with some data fixed), which we regard as the 'moduli space' of Buchberger algorithms.

In this paper, we are going to consider the space of polynomials $X = X(n, r, \{P_i\}_{i=1}^r)$, where n, r are positive integers and $\{P_i\}_{i=1}^r$ a collection of finite sets of monomials, which parametrizes all sequences $F = (F_1, \ldots, F_r)$ of n-variable polynomials such that each F_i has the monomials in P_i. Our object of study is the partition $\mathscr{S} = \{ S_\lambda \}_{\lambda \in \Lambda}$ of this space induced from the notion of *G-types* of $F = (F_1, \ldots, F_r)$ (§2.6). Roughly, the G-type collects all *types* of the polynomials that appear in the whole process of the Buchberger algorithm (with any term order fixed once for all), where by the type of a polynomial we mean the set of all monomials with non-zero coefficients. In other words, G-types drop off all the coefficients, but keeping the knowledge about whether being zero or non-zero, of

the polynomials that appear in the process, and thus provide a feasible way to encode the 'process type' of the algorithm. This gives a partition of the space X into at most countably many subsets, called *G-isotypic* loci. An easy, but perhaps requiring a little care, fact. is that each G-isotypic locus is actually a locally closed subspace of X (Proposition 3.3). This is of course not a surprising result, but combined with a simple observation from algebraic geometry (Lemma 3.4), one can show that the number of the subsets in $\mathscr{S} = \{\, S_\lambda \,\}_{\lambda \in \Lambda}$ is actually finite (Theorem 3.5); in other words, once the data n, r and $\{P_i\}$ as above are fixed, then there are only finitely many possible types of the flow charts, i.e., G-types, of the Buchberger processes starting from the sequences of polynomials in X, which seems highly non-trivial. We thus obtain a finite stratification $\mathscr{S} = \{\, S_\lambda \,\}_{\lambda \in \Lambda}$ on the space X by G-isotypic loci. We call this stratification the *Buchberger stratification.*

As a corollary, one immediately sees that the number (resp. the maximal degree) of the polynomials that appear in the Gröbner basis of the ideal generated by $F_1, \ldots, F_r$ from X is upper-bounded by a universal constant that only depends on the data n, r, $\{P_i\}$, and the term ordering (Corollary 3.8). Unfortunately, our method cannot give the explicit bound, but could give the existence in a quite elementary way.

In the final section, we will consider a simple and explicit example, and enumerate all the G-types therein.

After submiting the first draft of this paper, the authors are informed from the referee that what we call Buchberger stratification in this paper, or at least a notion similar to it, has already been developed by Weispfenning [5]; see [3] for more information. In spite of this similarity, we think that our scope of this paper differs from these works, mainly in that we regard the existence of the stratifications as one of the most important geometric backgrounds for the existence of the upper-bounds; notice that the key for the bounded-ness lies not in the computation-theoretic aspects, but rather in the *geometric* backgrounds, as the simple algebraic-geometry lemma (Lemma 3.4) shows.

2 G-types

2.1 Notation and conventions

Let $n \geq 1$ be a positive integer, and consider the ring $\mathbb{C}[X_1, \ldots, X_n]$ of polynomials of n variables. We will employ the multi-index notation as follows: For any $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, where $\mathbb{N} = \mathbb{Z}_{\geq 0}$ is the set of non-negative integers, we write $X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n}$ and $|\nu| = \nu_1 + \cdots + \nu_n$; each polynomial $F = F(X) \in \mathbb{C}[X_1, \ldots, X_n]$ can be written as $F = \sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu$, where $a_\nu = 0$ for all but finitely many ν's.

We fix once for all a term order $\leq$ on $\mathbb{N}^n$. For any non-zero $F \in \mathbb{C}[X_1, \ldots, X_n]$,

we denote by $\nu(F) = (\nu(F)_1, \ldots, \nu(F)_n)$, $\mathrm{LT}(F)$, and $\mathrm{LM}(F) = X^{\nu(F)}$, the leading degree, the leading term, and the leading monomial, respectively, of F.

For a non-zero $F \in \mathbb{C}[X_1, \ldots, X_n]$, we denote by $C_F = \nu(F) + \mathbb{N}^n$ the cone in $\mathbb{N}^n$ generated by the leading degree of F. For an ordered sequence $\boldsymbol{F} = (F_1, \ldots, F_r)$ of non-zero polynomials, we set $C_{\boldsymbol{F}} = \bigcup_{i=1}^{r} C_{F_i}$.

2.2 Type of polynomials

For a polynomial $F = \sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu \in \mathbb{C}[X_1, \ldots, X_n]$, the *type* of F is defined to be the finite set

$$\tau(F) = \{\, \nu \in \mathbb{N}^n \mid a_\nu \neq 0 \,\}.$$

Clearly, we have:

- $F = 0$ if and only if $\tau(F) = \emptyset$;

- if $F \neq 0$, its degree $\deg(F)$ is equal to $\max\{\, |\nu| \mid \nu \in \tau(F) \,\}$.

Two polynomials $F, G \in \mathbb{C}[X_1, \ldots, X_n]$ are said to be *isotypic*, written $F \overset{\tau}{\sim} G$, if $\tau(F) = \tau(G)$. It is clear that the relation $\overset{\tau}{\sim}$ is an equivalence relation on the set $\mathbb{C}[X_1, \ldots, X_n]$.

The *type* of an ordered set $\boldsymbol{F} = (F_1, \ldots, F_r)$ of polynomials is defined to be

$$\tau(\boldsymbol{F}) = (\tau(F_1), \ldots, \tau(F_r)),$$

and two ordered sets $\boldsymbol{F}, \boldsymbol{G}$ of polynomials are said to be *isotypic*, written $\boldsymbol{F} \overset{\tau}{\sim} \boldsymbol{G}$, if $\tau(\boldsymbol{F}) = \tau(\boldsymbol{G})$.

2.3 Division algorithm

Let $\boldsymbol{F} = (F_1, \ldots, F_r)$ be an ordered set of non-zero polynomials in $\mathbb{C}[X_1, \ldots, X_n]$, and $H \in \mathbb{C}[X_1, \ldots, X_n]$ a polynomial. We say H is $\boldsymbol{F}$-*reduced* if $\tau(H) \cap C_{\boldsymbol{F}} = \emptyset$. For any $H \in \mathbb{C}[X_1, \ldots, X_n]$ we can always find $Q_1, \ldots, Q_r \in \mathbb{C}[X_1, \ldots, X_n]$ such that $R = H - (Q_1 F_1 + \cdots + Q_r F_r)$ is $\boldsymbol{F}$-reduced. Such an R is, however, not in general uniquely determined by H. Hence, for the sake of uniformity, we need to fix an explicit process to obtain it.

To do this, let us illustrate a single division step as follows: If H is not $\boldsymbol{F}$-reduced, let $\nu_0 = \max \tau(H) \cap C_{\boldsymbol{F}}$ be the maximal (with respect to the fixed term order $\leq$) degree in $\tau(H) \cap C_{\boldsymbol{F}}$, and take the minimal $i = i_0$ in the set $\{\, i \mid \nu_0 \in C_{F_i} \,\}$. Then we set

$$H' = H - \frac{\mathrm{T}_{\nu_0}(H)}{\mathrm{LT}(F_{i_0})} F_{i_0}, \qquad (*)$$

where $\mathrm{T}_{\nu_0}(H)$ is the degree-ν_0 term of H. This process will be denoted by $H \overset{\boldsymbol{F}}{\rightsquigarrow} H'$, and the pair (ν_0, i_0) as above will be called the *initial index*. One can iterate this process until we get the $\boldsymbol{F}$-reduced result in the end (cf. [1, §1.5]):

$$H = H_0 \overset{\boldsymbol{F}}{\rightsquigarrow} H_1 \overset{\boldsymbol{F}}{\rightsquigarrow} \cdots \overset{\boldsymbol{F}}{\rightsquigarrow} H_s, \qquad (**)$$

often written $H \xrightarrow{F}_{+} R$ for short, where $R = H_s$ is F-reduced. We define the *division type* of H with respect to F to be

$$\tau_D(H, F) = (\tau(F); \tau(H_0), \tau(H_1), \ldots, \tau(H_s)),$$

referring to the diagram (∗∗).

2.4 S-polynomial

For two non-zero polynomials $F, G \in \mathbb{C}[X_1, \ldots, X_n]$, the so-called S-polynomial of F and G is defined as follows: Define $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ by $\mu_i = \max \{ \nu(F)_i, \nu(G)_i \}$ for $i = 1, \ldots, n$. Then set

$$S(F, G) = \frac{X^\mu \cdot F}{\mathrm{LT}(F)} - \frac{X^\mu \cdot G}{\mathrm{LT}(G)}.$$

If an ordered set $F = (F_1, \ldots, F_r)$ of non-zero polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ is given, one has the F-reduction of $S(F, G)$, which we denote by $\overline{S}_F(F, G)$, i.e.,

$$S(F, G) \xrightarrow{F}_{+} \overline{S}_F(F, G)$$

in the notation as in §2.3.

2.5   Gröbner basis

A sequence $F = (F_1, \ldots, F_r)$ of non-zero polynomials is called a *Gröbner basis* (of the ideal generated by $F_1, \ldots, F_r$) if, for any $i, j = 1, \ldots, n$ with $i < j$, we have $\overline{S}_F(F_i, F_j) = 0$. If $F = (F_1, \ldots, F_r)$ is not a Gröbner basis, one has (i, j) such that $\overline{S}_F(F_i, F_j) \neq 0$. Take such (i_0, j_0) that is minimal in the set $\{ (i, j) \mid i, j = 1, \ldots, r, \ i < j \}$ with respect to the lexicographical ordering, and define a new sequence of polynomials $F' = (F_1, \ldots, F_{r+1})$ by

$$F_{r+1} = \overline{S}_F(F_{i_0}, F_{j_0}).$$

Let us denote this process by $F \xrightarrow{B} F'$, and call the index (i_0, j_0) as above the *initial index*; if F is a Gröbner basis, we set $(i_0, j_0) = \infty$. A classical but significant fact is that, starting from arbitrary $F = F_0 = (F_1, \ldots, F_r)$, one eventually obtains a Gröbner basis F_s by iterating the above process (cf. [1, §1.7]):

$$F = F_0 \xrightarrow{B} F_1 \xrightarrow{B} \cdots \xrightarrow{B} F_s, \qquad\qquad (*)$$

often written $F_0 \xrightarrow{B}_{+} F_s$ for short.

2.6   G-type

Consider the single step $F \xrightarrow{B} F'$ as above. We define the *B-type* of F to be the following data:

$$\tau_B(F) = (\tau_D(S(F_i, F_j), F))_{(i,j) \leq (i_0, j_0)},$$

where (i, j) runs over the set $\{ (i, j) \mid i, j = 1, \ldots, r, \ i < j, \ (i, j) \le (i_0, j_0) \}$ (ordered by the lexicographical order) and (i_0, j_0) is the initial index of F. Finally, referring to the diagram $(*)$ in §2.5, we define the *G-type* of F to be

$$\tau_{\mathrm{G}}(F) = (\tau_{\mathrm{B}}(F_0), \ldots, \tau_{\mathrm{B}}(F_s)).$$

Two ordered sequences $F = (F_1, \ldots, F_r)$ and $G = (G_1, \ldots, G_s)$ are said to be *G-isotypic*, written $F \overset{\mathrm{G}}{\sim} G$, if we have $\tau_{\mathrm{G}}(F) = \tau_{\mathrm{G}}(G)$. It is clear that the relation $\overset{\mathrm{G}}{\sim}$ is an equivalence relation on the set of all ordered sequences of non-zero polynomials. Notice that $F \overset{\mathrm{G}}{\sim} G$ in particular requires that $\tau(F) = \tau(G)$, hence that $r = s$ and $\tau(F_i) = \tau(G_i)$ for any $i = 1, \ldots, r$.

3 Buchberger stratification

3.1 The space of polynomials

Let $r \ge 1$ be a positive integer, and $P_1, \ldots, P_r$ non-empty finite subsets of $\mathbb{N}^n$. The typical choices of P_i's are as follows:

(1) (Inhomogenous case) let $d_1, \ldots, d_r$ be a sequence of non-negative integers and $P_i = \{ \nu \in \mathbb{N}^n \mid |\nu| \le d_i \}$ for $i = 1, \ldots, r$;

(2) (Homogenous case) let $d_1, \ldots, d_r$ be a sequence of non-negative integers and $P_i = \{ \nu \in \mathbb{N}^n \mid |\nu| = d_i \}$ for $i = 1, \ldots, r$.

Let V_i for $i = 1, \ldots, r$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}[X_1, \ldots, X_n]$ spanned by $\{ X^\nu \mid \nu \in P_i \}$. Consider the associated projective space $\mathbb{P}(V_i)$ for each $i = 1, \ldots, r$; each point of $\mathbb{P}(V_i)$ corresponds to a non-zero polynomial F_i, considered up to multiplication by non-zero constants, such that $\tau(F_i) \subseteq P_i$. Set

$$X(n, r, \{P_i\}_{i=1}^r) = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_r),$$

whose points are in one to one correspondence with ordered r-tuples $F = (F_1, \ldots, F_r)$ of non-zero polynomials, each considered up to multiplication by non-zero constants, such that $\tau(F_i) \subset P_i$ for $i = 1, \ldots, r$.

3.2 Subsets by G-types

The equivalence relation $\overset{\mathrm{G}}{\sim}$ defined in §2.6 gives a set-theoretic partition $\mathscr{S} = \{ S_\lambda \}_{\lambda \in \Lambda}$ of $X = X(n, r, \{P_i\}_{i=1}^r)$ by G-isotypic classes, where the set Λ indexes all possible G-types of ordered r-tuples $F = (F_1, \ldots, F_r)$ corresponding to points of X.

3.3 Proposition

The subset S_λ for each $\lambda \in \Lambda$ is a locally closed set in the projective variety $X = X(n, r, \{P_i\}_{i=1}^r)$ with respect to the Zariski topology, i.e., the intersection of a Zariski closed subset and a Zariski open subset.

Proof. We may suppose $S_\lambda \neq \emptyset$. Take $F = (F_1, \ldots, F_r) \in S_\lambda$, and write $F_i = \sum_{\nu \in \tau(F_i)} a_{i,\nu} X^\nu$ for $i = 1, \ldots, r$; notice that coefficients $a_{i,\nu}$ are all non-zero. Any point of X corresponds to $\widetilde{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_r)$ with $\widetilde{F}_i = F_i + \sum_{\nu \in P_i} \xi_{i,\nu} X^\nu$ for $i = 1, \ldots, r$. We need to show that the condition for $\widetilde{F}$ to have the same G-type as F is given by finitely many zero's and non-zero's of polynomials of $\{\xi_{i,\nu}\}$, i.e.,

(*) there exist a finite set of polynomials $\{\Phi_\alpha\}_\alpha$ and a set of finitely many non-zero polynomials $\{\Psi_\beta\}_\beta$ of $\{\xi_{i,\nu}\}$ such that the condition is given by $\Phi_\alpha(\xi_{i,\nu}) = 0$ for all α and $\Psi_\beta(\xi_{i,\nu}) \neq 0$ for all β.

Notice that the condition $\tau(F) = \tau(\widetilde{F})$ already requires $\xi_{i,\nu} = 0$ for $\nu \notin \tau(F_i)$. In order for the following inductive argument to make sense, we need to put $\widetilde{F}_i$ to have more general form: $\widetilde{F}_i = \sum_{\nu \in \tau(F_i)} \phi_{i,\nu}(\xi) X^\nu$ for $i = 1, \ldots, r$, where $\phi_{i,\nu}(\xi)$ is a non-zero rational function of $\xi_{i,\nu}$'s such that $\phi_{i,\nu}(0) = a_{i,\nu} \neq 0$.

Set $\tau_G(F) = (\tau_B(F_0), \ldots, \tau_B(F_s))$ and $\tau_G(\widetilde{F}) = (\tau_B(\widetilde{F}_0), \ldots, \tau_B(\widetilde{F}_{\tilde{s}}))$, and we will always indicate by tilde the corresponding objects constructed from $\widetilde{F}$. Our condition $\tau_G(F) = \tau_G(\widetilde{F})$ especially implies $s = \tilde{s}$. If we show that the first condition $\tau_B(F_0) = \tau_B(\widetilde{F}_0)$ is equivalent to a condition of the form (*) above, then the proposition follows by induction with respect to s (since $\tau_B(F_0) = \tau_B(\widetilde{F}_0)$ implies that $\tau(F_1) = \tau(\widetilde{F}_1)$). Consider, then, the condition $\tau_B(F) = \tau_B(\widetilde{F})$ (under the hypothesis $\tau(F) = \tau(\widetilde{F})$). If we have $\tau_D(S(F_i, F_j), F) = \tau_D(S(\widetilde{F}_i, \widetilde{F}_j), \widetilde{F})$ for all $(i,j) < (i_0, j_0)$, then it follows, in particular, that the initial index (i_0, j_0) of F is equal to the initial index of $\widetilde{F}$. Hence it suffices to show that the condition $\tau_D(S(F_i, F_j), F) = \tau_D(S(\widetilde{F}_i, \widetilde{F}_j), \widetilde{F})$ is equivalent to a condition of the form (*) above for any $(i,j) \leq (i_0, j_0)$.

Therefore, the proof boils down to showing the following claims: Let F and $\widetilde{F}$ be as above, and $H = \sum_\nu b_\nu X^\nu$ and $\widetilde{H} = \sum_{\nu \in \tau(H)} \psi_\nu(\xi) X^\nu$, where $\psi_\nu(\xi)$ is a rational function of $\xi_{i,\nu}$'s such that $\psi_\nu(0) = b_\nu \neq 0$; then

(a) the coefficients of $\tau(S(\widetilde{F}_i, \widetilde{F}_j))$ of each degree ν is a rational function η_{ij} of $\xi_{i,\nu}$'s; if $\nu \in \tau(S(F_i, F_j))$, then $\eta_{ij}(0)$ is equal to the degree-ν coefficient of $S(F_i, F_j)$, hence non-zero;

(b) if $H \xrightarrow{F} H'$ and $\widetilde{H} \xrightarrow{\widetilde{F}} \widetilde{H}'$, then the coefficients of $\widetilde{H}'$ of each degree ν is a rational function ζ_{ij} of $\xi_{i,\nu}$'s; if $\nu \in \tau(H')$, then $\zeta_{ij}(0)$ is equal to the degree-ν coefficient of H', hence non-zero.

Notice that, in (b), the initial indices of $H \xrightarrow{F} H'$ and $\widetilde{H} \xrightarrow{\widetilde{F}} \widetilde{H}'$ must be the same, since $\tau(H) = \tau(\widetilde{H})$ and $\tau(F) = \tau(\widetilde{F})$. Both assertions (a) and (b) can be verified by an easy calculation, and hence the proof is done.

Thus the G-types give a partition $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ of $X = X(n, r, \{P_i\}_{i=1}^r)$ by locally closed subspaces. On the other hand, there are only countably many possible G-types that arise from X. The following easy lemma shows that, in fact, the partition is finite:

3.4 Lemma

Let X be an algebraic variety over $\mathbb{C}$, and $X = \coprod_{\lambda \in \Lambda} S_\lambda$ a countable partition of X by locally closed subsets. Then the partition is finite; i.e., there exists a finite set $\{\lambda_1, \ldots, \lambda_r\} \subseteq \Lambda$ such that $X = \coprod_{i=1}^r S_{\lambda_i}$.

Proof. We show the lemma by induction with respect to the maximal dimension n of the irreducible components of X. If $n = 0$, then X consists of finitely many points, and the claim is trivial. In general, we may assume that X is irreducible. Moreover, considering the irreducible decomposition of each S_λ, we may assume that every S_λ is irreducible. Since the complement of any non-empty open subset consists of irreducible components of dimension $< n$, we may assume that X is affine. Suppose there exist no S_λ that is open in X. Then every S_λ is an irreducible subvariety of X of dimension $< n$. By Noether's normalization theorem, we have a finite surjective morphism $\pi \colon X \to \mathbb{A}^n$ to the affine n-space. It follows that the countably many subvarieties $\pi(S_\lambda)$ of dimension $< n$ cover the affine space $\mathbb{A}^n$, which is absurd. Hence there must exist an open S_λ. By induction, the complement $X \setminus S_\lambda$ is covered by finitely many S_μ's, which shows the claim.

3.5 Theorem

The partition $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ gives a stratification on the variety $X = X(n, r, \{P_i\}_{i=1}^r)$ by finitely many locally closed subspaces. $\square$

We call this stratification the *Buchberger stratification*.

3.6 Remark

The theorem shows that there exists a unique Zariski open dense stratum among S_λ's, which implies that there is a unique process type of the algorithm that applies to general sequences $F = (F_1, \ldots, F_r)$ of polynomials. The process thus characterized should be appropriately called the *generic process* with respect to n, r, and $\{P_i\}$.

3.7 Corollary

For fixed n (= the number of variables), r (= the number of polynomials), and non-empty finite subsets $P_1, \ldots, P_r$, there are only finitely many possible G-types of $F = (F_1, \ldots, F_r) \in X$. $\square$

Since the number and the degrees of polynomials that appear in the Buchberger algorithm are preserved by passage to their types, one has in particular the following:

3.8 Corollary

There exists a number $N = N(n, r, \{P_i\}, \leq)$ (resp. $D = D(n, r, \{P_i\}, \leq)$), depending only on n, r, $\{P_i\}$, and the term ordering $\leq$ such that, for any $F =$

$(F_1, \ldots, F_r) \in X = X(n, r, \{P_i\}_{i=1}^r)$, the number (resp. the maximal degree) of the polynomials that appear in the Gröbner basis of the ideal generated by $F_1, \ldots, F_r$ is upper-bounded by N (resp. D). □

4 An example

4.1

Let us consider the case $n = r = 2$, and $P_1 = \{(2,0), (0,2)\}$, $P_2 = \{(2,0), (1,1)\}$, with the lexicographical ordering $\leq$. The sequences $F = F_0$ in question are of the form $F = (F_1 = a_1 x^2 + c_1 y^2, F_2 = a_2 x^2 + b_2 xy)$, where we write $x = X_1$ and $y = X_2$. Notice that the corresponding space $X = X(2, 2, \{P_i\})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with bi-homogenous coordinate $(a_1 : c_1) \times (a_2 : b_2)$.

4.2

First, we assume that the coefficients a_1, c_1, a_2, b_2 are all non-zero, that is, $\tau(F_0) = (\{(2,0), (0,2)\}, \{(2,0), (1,1)\})$. We have $\tau_{\mathrm{D}}(S(F_1, F_2), F_0) = (\tau(F_0; \{(1,1), (0,2)\}))$, $\tau_{\mathrm{B}}(F_0) = (\tau_{\mathrm{D}}(S(F_1, F_2), F_0))$, and $F_3 = \overline{S}_{F_0}(F_1, F_2) = S(F_1, F_2) = -b_3 xy + c_3 y^2$, where $b_3 = b_2/a_2 \neq 0$ and $c_3 = c_1/a_1 \neq 0$. Then we set $F_1 = (F_1, F_2, F_3)$. Since $S(F_1, F_2) = F_3$ reduces to 0 by F_1, we have $\tau_{\mathrm{D}}(S(F_1, F_2), F_1) = (\tau(F_1); \{(1,1), (0,2)\}, \emptyset)$. Since

$$S(F_1, F_3) = \frac{c_3}{b_3} xy^2 + \frac{c_1}{a_1} y^3 \overset{F_1}{\rightsquigarrow} \left(\frac{c_1}{a_1} + \frac{c_3^2}{b_3^2}\right) y^3,$$

we have two possibilities

$$\tau_{\mathrm{D}}(S(F_1, F_3), F_1) = \begin{cases} (\tau(F_1); \{(1,2), (0,3)\}, \{(0,3)\}) & \text{if } a_1 b_2^2 + c_1 a_2^2 \neq 0, \\ (\tau(F_1); \{(1,2), (0,3)\}, \emptyset) & \text{if } a_1 b_2^2 + c_1 a_2^2 = 0. \end{cases}$$

In both cases, we have $\tau_{\mathrm{B}}(F_1) = (\tau_{\mathrm{D}}(S(F_1, F_2), F_1), \tau_{\mathrm{D}}(S(F_1, F_3), F_1))$.

(1) If $a_1 b_2^2 + c_1 a_2^2 \neq 0$, then we set $F_4 = c_4 y^3$, where $c_4 = c_1/a_1 + c_3^2/b_3^2 \neq 0$, and $F_2 = (F_1, F_2, F_3, F_4)$. We have $\tau_{\mathrm{D}}(S(F_1, F_2), F_2) = (\tau(F_2); \{(1,1), (0,2)\}, \emptyset)$, $\tau_{\mathrm{D}}(S(F_1, F_3), F_2) = (\tau(F_2); \{(1,2), (0,3)\}, \{(0,3)\}, \emptyset)$, and $\tau_{\mathrm{D}}(S(F_1, F_4), F_2) = (\tau(F_2); \{(0,5)\}, \emptyset)$. Since

$$S(F_2, F_3) = \left(\frac{b_2}{a_2} + \frac{c_3}{b_3}\right) xy^2 \overset{F_2}{\rightsquigarrow} \left(\frac{b_2}{a_2} + \frac{c_3}{b_3}\right) \frac{c_3}{b_3} y^3 \overset{F_2}{\rightsquigarrow} 0,$$

and $b_2/a_2 + c_3/b_3 = (a_1 b_2^2 + c_1 a_2^2)/a_1 a_2 b_2 \neq 0$, we have

$$\tau_{\mathrm{D}}(S(F_2, F_3), F_2) = (\tau(F_2); \{(1,2)\}, \{(0,3)\}, \emptyset).$$

Then one goes on calculating to obtain

$$\tau_{\mathrm{D}}(S(F_2, F_4), F_2) = (\tau(F_2); \{(1,4)\}, \{(0,5)\}, \emptyset),$$
$$\tau_{\mathrm{D}}(S(F_3, F_4), F_2) = (\tau(F_2); \{(0,4)\}, \emptyset),$$

whereby concluding that F_2 is the Gröbner basis. Thus we have exactly one G-type in this case.

(2) If $a_1 b_2^2 + c_1 a_2^2 = 0$, then we proceed to $S(F_2, F_3)$; since

$$S(F_2, F_3) = \left(\frac{b_2}{a_2} + \frac{c_3}{b_3} \right) xy^2 = 0,$$

we have

$$\tau_{\mathrm{D}}(S(F_2, F_3), F_1) = (\tau(F_1); \emptyset).$$

Then we go on calculating the other S-polynomials to see that F_1 is actually a Gröbner basis. Hence, we have exactly one G-type in this situation.

In total, we have the two G-types, which correspond to the Zariski open dense stratum $a_1 b_2^2 + c_1 a_2^2 \neq 0$, which corresponds to the generic process (cf. Remark 3.6), and the stratum given by $a_1 b_2^2 + c_1 a_2^2 = 0$.

4.3

If one (and only one) of a_1, c_1, a_2, b_2 in the bi-homogenous coordinate $(a_1 : c_1) \times (a_2 : b_2)$ is zero, then, by an easy calculation, one has the single possible G-type in each case, and hence we have in total four strata on X, each given by a line in $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. If two of a_1, c_1, a_2, b_2 are zero, then F_1 and F_2 are monomials and hence F_0 itself is a Gröbner basis. This comprises four G-types, which correspond to points of X.

Summing up all this, we conclude that the Buchberger stratification on X consist of 10 strata, one 2-dimensional (Zariski open strata), five 1-dimensional, and four 0-dimensional.

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