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# A remark on the range of three dimensional pinned random walks

*Dedicated to Professor Shin'ichi Kotani on his 60th birthday*

Yuji Hamana*

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**Abstract.** The range of a random walk means the number of distinct sites visited at least once by the random walk. In the three dimensional case, it has already known that the second term of the expectation of the range of the simple symmetric random walk under the conditional probability given the event that the last point is the origin is small in comparison with that of the original random walk. This paper claims that the second term in the pinned case is bounded.

## 1. Introduction

Asymptotic behavior of the expected volume of the Wiener sausage for a Brownian bridge on the time interval $[0,t]$ associated with a non-polar compact set was supplied by van den Berg and Bolthausen [1] in the two dimensional case and by McGillivray [5] in higher dimensional cases. They conclude that the leading term is the same as that of the Wiener sausage for a Brownian motion up to time $t$, which is given by Spitzer [6]. In the case that the non-polar compact set is the closed ball with radius $r$, the second term of the expected volume of the pinned Wiener sausage is small in comparison with the non-pinned Wiener sausage if the dimension is three. McGillivray [5] showed that the former is $6\pi r^3$, which is already implicit in the work by Uhlenbeck and Beth [7], and Le Gall [4] showed that the latter is $4\sqrt{2\pi}r^2\sqrt{t}$.

A discrete analogue of the volume of the Wiener sausage up to time $t$ is the number of distinct sites entered by a random walk in the first $n$ steps, which is called the range at time $n$ of the random walk. Asymptotic behavior of the expectation of the range at time $n$ of the simple random walk was first given by Dvoretzky and Erdős [2], and by Hamana [3] in the pinned case. Similarly to the Wiener sausage, their results show that each expectation has the same leading

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term, and if the dimension is three, Hamana proved that the second term of the expectation of the range of the non-pinned simple random walk is $c\sqrt{n}$ for some suitable positive constant $c$ and that of the pinned simple random walk is of order $\sqrt{n}(\log n)^{-\delta}$ for any $\delta > 0$. The result for the Wiener sausage shows that the last result can stand further improvement. The conclusion in this paper is that the second term of the expectation of the range of simple random walk is bounded for dimension three.

## 2. Preliminaries and notation

By a random walk $\{S_n\}_{n=0}^{\infty}$ on the $d$ dimensional integer lattice $\mathbb{Z}^d$, we mean a sequence of random variables defined by $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables which take values in $\mathbb{Z}^d$. The simple random walk means a random walk such that $P[X_1 = x]$ is equal to $1/2d$ if $x$ is a unit vector in $\mathbb{Z}^d$ and is equal to 0 otherwise. Throughout this paper we consider the $d$ dimensional simple random walk. Let $\gamma_d$ be the probability that a random walk never returns to the starting point. It is well-known that $\gamma_d$ is strictly positive if $d \geqq 3$ and equal to 0 otherwise.

Since it will be convenient to regard the random walk as a Markov chain, we will use some terminology of general Markov chains. For $x \in \mathbb{Z}^d$ let $P_x[\,\cdot\,]$ denote the probability measures of events related to the random walk starting at $x$. When $x = 0$, we simply write $P[\,\cdot\,]$ instead of $P_0[\,\cdot\,]$. For $n \geqq 0$ and $x, y \in \mathbb{Z}^d$ the notation $p^n(x,y)$ is used for $P_x[S_n = y]$. It is obvious that $p^n(x,y) = p^n(0, y-x)$. For $x \in \mathbb{Z}^d$ let $\tau_x$ be the first hitting time of $x$, that is, $\tau_x = \inf\{n \geqq 1\,;\, S_n = x\}$. If there are no positive integers with $S_n = x$, then $\tau_x = \infty$. The taboo probabilities are defined by

$$p_z^n(x,y) = P_x[S_n = y,\ \tau_z \geqq n].$$

A simple calculation shows that

$$p_0^n(0,x) = p_x^n(0,x) \tag{2.1}$$

for all $n \geqq 1$ and $x \in \mathbb{Z}^d$.

For simplicity we will use $u_n$ for $p^n(0,0)$ and $f_n$ for $p_0^n(0,0)$. If $n$ is odd, both $u_n$ and $f_n$ are equal to 0. It is well-known that

$$u_{2n} = \sum_{k=1}^{n-1} f_{2k} u_{2(n-k)} + f_{2n}. \tag{2.2}$$

Another useful formula is that

$$u_{2n} = \kappa_d n^{-d/2} + O[n^{-1-d/2}], \tag{2.3}$$

where $\kappa_d = 2(d/4\pi)^{d/2}$. For sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ of real numbers such that $c_n > 0$ for $n \geqq 1$, the notation $a_n = b_n + O[c_n]$ means that $(a_n - b_n)/c_n$ remains bounded.

Throughout this paper, $C_1, C_2, \cdots, C_{11}$ will denote suitable positive real constants.

## 3. Main result and proof

For a positive integer $n$ let

$$R_n = |\{S_1, S_2, \cdots, S_n\}|,$$

where $|A|$ denotes the cardinality of a set $A$. We call $R_n$ the range at time $n$ of the random walk. If $d = 3$, Hamana [3] proved that

$$E[R_{2n}|S_{2n} = 0] = 2\gamma_3 n + O[n^{1/2}(\log n)^{-\delta}] \tag{3.1}$$

for any $\delta > 0$. However, this result must be improved when we consider an analogy between random walk and Brownian motion. The mail result in this paper is the following.

**Theorem 3.1.** *If $d = 3$, we have that*

$$E[R_{2n}|S_{2n} = 0] = 2\gamma_3 n + O[1]. \tag{3.2}$$

Unfortunately we have no idea for calculating the explicit form of the second term of the right hand side of (3.2).

We now give a proof of Theorem 3.1. Since we treat only the three dimensional random walk in this section, we write $\gamma$ and $\kappa$ for $\gamma_3$ and $\kappa_3$, respectively for simplicity. According to (3.21) in Hamana [3], we have that $E[R_{2n}|S_{2n} = 0]$ is equal to

$$2\gamma n + 2n \sum_{h=n}^{\infty} f_{2h} - 2n^{3/2} \sum_{h=1}^{n-1} \left(h^{-1/2} - n^{-1/2}\right) f_{2(n-h)} + O[1]. \tag{3.3}$$

It suffices to give asymptotic behavior of $f_{2n}$ for calculation of (3.3). The second claim of Lemma 3.1 in Hamana [3] is that

$$f_{2n} = \gamma^2 \kappa n^{-3/2} + O[n^{-3/2}(\log n)^{-\delta}]$$

for any $\delta > 0$. Since this formula can provide only (3.1), we need to improve this estimate in order to show (3.2). We accordingly obtain the following lemma, of which the proof is defered to Section 4.

**Lemma 3.2.** *We have that*

$$f_{2n} = \gamma^2 \kappa n^{-3/2} + O[n^{-17/8}].$$

*for the three dimensional simple random walk.*

In virtue of this lemma, we obtain that the second term of (3.3) is

$$4\gamma^2\kappa n^{1/2} + O[n^{-1/8}].$$

We can calculate the third term of (3.3) in an analogous manner to (3.23) in Hamana [3]. It follows from Lemma 3.2 that

$$\sum_{h=1}^{n-1}\left(h^{-1/2} - n^{-1/2}\right) f_{2(n-h)} \tag{3.4}$$

is equal to

$$\frac{\gamma^2\kappa}{\sqrt{n}}\sum_{h=1}^{n-1}\frac{1}{\sqrt{h(n-h)}(\sqrt{h}+\sqrt{n})} + O\left[\frac{1}{\sqrt{n}}\sum_{h=1}^{n-1}\frac{1}{\sqrt{h}(n-h)^{9/8}(\sqrt{h}+\sqrt{n})}\right]. \tag{3.5}$$

Since the first term of (3.5) is the same as (3.25) in Hamana [3], we have that it is

$$2\gamma^2\kappa n^{-1} + O[n^{-3/2}].$$

It suffices for an estimate of the second term of (3.5) to give a bound of

$$\frac{1}{n}\sum_{h=1}^{n-1}\frac{1}{\sqrt{h}(n-h)^{9/8}}.$$

Since it is of order $n^{-3/2}$, we have that (3.4) is $2\gamma^2\kappa n^{-1} + O[n^{-3/2}]$, which implies that the third term of (3.3) is

$$-4\gamma^2\kappa n^{1/2} + O[1].$$

This completes the proof of Theorem 3.1.

**Remark.** In the three dimensional case, Proposition 2.1 in Hamana [3] shows that

$$ER_n = \gamma n + 2^{5/2}\gamma^2\kappa n^{1/2} + O[n^{1/2}(\log n)^{-\delta}]$$

for any $\delta > 0$. With the help of Lemma 3.2, the estimate of the error term can be easily improved. We then conclude that

$$ER_n = \gamma n + 2^{5/2}\gamma^2\kappa n^{1/2} + O[1].$$

## 4. Proof of Lemma 3.2

This section is devoted to a proof of Lemma 3.2. We also consider the simple random walk moving on $\mathbb{Z}^3$.

Let $N = [n/4]$ for a positive integer $n$, where the notation $[x]$ is used for the greatest integer which is not larger than a real number $x$. Note that

$$f_{2n} = \sum_{x\neq 0}\sum_{y\neq 0} p_0^{2N}(0,x)p_0^{2n-4N}(x,y)p_0^{2N}(y,0) \tag{4.1}$$

for a positive integer $n$. For simplicity, we use $L$ for $n-2N$. We first consider the effect of replacing $p_0^{2L}(x,y)$ with $p^{2L}(x,y)$ in (4.1), for which the following equality will be useful:

$$p^{2L}(x,y) - p_0^{2L}(x,y) = P_x\left[\tau_0 \leqq L, S_{2L}=y\right] + P_x\left[L < \tau_0 \leqq 2L, S_{2L}=y\right]. \quad (4.2)$$

It is easy to see the the first term of (4.2) is equal to

$$\sum_{k=1}^{L} p_0^k(x,0)p^{2L}(0,y).$$

Classifying the event $\{L < \tau_0 \leqq 2L, S_{2L}=y\}$ by the last hitting time of 0, we can obtain that the second term of the right hand side of (4.2) is equal to

$$\sum_{k=L+1}^{2L-1} P_x[\sigma_0 = k,\ S_{2L}=y] - \sum_{j=L+1}^{2L-1}\sum_{k=1}^{L} P_x[\tau_0 = k,\ \sigma_0 = j,\ S_{2L}=y], \quad (4.3)$$

where $\sigma_0 = \max\{\alpha \leqq L\,;\ S_{2\alpha}=0\}$. The first term of (4.3) is expressed by

$$\sum_{k=L+1}^{2L-1} p^k(x,0)p_0^{2L-k}(0,y)$$

and the second one is expressed by

$$-\sum_{j=L+1}^{2L-1}\sum_{k=1}^{L} p_0^k(x,0)p^{j-k}(0,0)p_0^{2L-j}(0,y).$$

Recall that $L = n-2N$. We hence have that $f_{2n}$ is equal to

$$\sum_{x\neq 0}\sum_{y\neq 0} p_0^{2N}(0,x)p^{2n-4N}(x,y)p_0^{2N}(y,0) \quad (4.4)$$

$$-\sum_{k=1}^{n-2N}\sum_{x\neq 0}\sum_{y\neq 0} p_0^{2N}(0,x)p_0^k(x,0)p^{2n-4N-k}(0,y)p_0^{2N}(y,0) \quad (4.5)$$

$$-\sum_{k=n-2N+1}^{2n-4N-1}\sum_{x\neq 0}\sum_{y\neq 0} p_0^{2N}(0,x)p^k(x,0)p_0^{2n-4N-k}(0,y)p_0^{2N}(y,0) \quad (4.6)$$

$$+\sum_{j=n-2N+1}^{2n-4N-1}\sum_{k=1}^{n-2N}\sum_{x\neq 0}\sum_{y\neq 0} p_0^{2N}(0,x)p_0^k(x,0)u_{j-k}p_0^{2n-4N-j}(0,y)p_0^{2N}(y,0). \quad (4.7)$$

We next try to show that

$$f_{2n} = u_{2n} - 2\sum_{k=1}^{[n/2]} f_{2k}u_{2n-2k} + \sum_{k=1}^{[n/2]}\sum_{j=1}^{[n/2]} f_{2k}f_{2j}u_{2n-2k-2j} + O[n^{-5/2}]. \quad (4.8)$$

We need the following lemma.

**Lemma 4.1.**

$$\sum_{w\neq 0} p_0^{2\alpha}(0,w)p^{2\beta}(w,z) = p^{2\alpha+2\beta}(0,z) - \sum_{k=1}^{\alpha} f_{2k}p^{2\alpha+2\beta-2k}(0,z). \tag{4.9}$$

**Proof.** If $\alpha = 1$, we can easily obtain (4.9) from the fact that $p_0^2(0,w) = p^2(0,w)$ for any $w$. We may consider the case when $\alpha \geqq 2$. For $w \neq 0$

$$p_0^{2\alpha}(0,w) = p^{2\alpha}(0,w) - \sum_{k=1}^{\alpha-1} f_{2k}p^{2\alpha-2k}(0,w).$$

Then

$$\begin{aligned}\sum_{w\neq 0} p_0^{2\alpha}(0,w)p^{2\beta}(w,z) =& p^{2\alpha+2\beta}(0,z) - p^{2\alpha}(0,0)p^{2\beta}(0,z)\\ &- \sum_{k=1}^{\alpha-1} f_{2k}p^{2\alpha+2\beta-2k}(0,z) + \sum_{k=1}^{\alpha-1} f_{2k}p^{2\alpha-2k}(0,0)p^{2\beta}(0,z).\end{aligned}$$

Using (2.2), we have that the forth term of the right hand side is equal to

$$u_{2\alpha}p^{2\beta}(0,z) - f_{2\alpha}p^{2\beta}(0,z),$$

which implies (4.9). □

An immediate consequence of (2.1) and (4.9) is that

$$\begin{aligned}\sum_{x\neq 0} p^{2\alpha}(0,x)p_0^{2\beta}(x,0) &= \sum_{w\neq 0} p^{2\alpha}(w,0)p_0^{2\beta}(0,w)\\ &= u_{2\alpha+2\beta} - \sum_{k=1}^{\beta} f_{2k}u_{2\alpha+2\beta-2k}.\end{aligned} \tag{4.10}$$

Lemma 4.1 yields that (4.4) is

$$\sum_{y\neq 0} p^{2n-2N}(0,y)p_0^{2N}(y,0) - \sum_{k=1}^{N}\sum_{y\neq 0} f_{2k}p^{2n-2N-2k}(0,y)p_0^{2N}(y,0).$$

Therefore it follows from (4.10) that (4.4) is equal to

$$u_{2n} - 2\sum_{k=1}^{N} f_{2k}u_{2n-2k} + \sum_{k=1}^{N}\sum_{j=1}^{N} f_{2k}f_{2j}u_{2n-2k-2j}.$$

For a calculation of (4.5), (4.6) and (4.7) we now consider the case that $n = 2m$. We have that (4.5) is equal to

$$-\sum_{k=1}^{n-2N} f_{2N+k}\sum_{y\neq 0} p^{2n-4N-k}(0,y)p_0^{2N}(y,0),$$

which can be expressed by

$$-\sum_{k=1}^{m-N} f_{2N+2k}\sum_{y\neq 0} p^{4m-4N-2k}(0,y)p_0^{2N}(y,0).$$

Making the substitution $h = N + k$ in the summation on $k$, we obtain that this summation coincides with

$$-\sum_{h=N+1}^{m} f_{2h}\sum_{y\neq 0} p^{4m-2N-2h}(0,y)p_0^{2N}(y,0). \tag{4.11}$$

It follows from (4.10) that (4.11) and also (4.5) are equal to

$$-\sum_{h=N+1}^{m} f_{2h}u_{4m-2h} + \sum_{h=N+1}^{m}\sum_{k=1}^{N} f_{2h}f_{2k}u_{4m-2h-2k}. \tag{4.12}$$

Similarly to (4.5), we can see that (4.6) has the following form:

$$-\sum_{k=m-N+1}^{2m-2N-1} f_{4m-2N-2k}\sum_{x\neq 0} p_0^{2N}(0,x)p^{2k}(x,0).$$

Substituting $h = 2m - N - k$ in the summation on $k$, we have that it is equal to

$$-\sum_{h=N+1}^{m-1} f_{2h}\sum_{x\neq 0} p_0^{2N}(0,x)p^{4m-2N-2h}(x,0). \tag{4.13}$$

Applying (4.9), we easily show that the difference between (4.12) and (4.13) is dominated by $f_{2m}u_{2m}$, which is of order $n^{-3}$. Therefore the leading term of (4.6) coincides with (4.12) and its remaining term is $O[n^{-3}]$.

It can be easily seen that (4.7) is of order $n^{-5/2}$. Indeed, it is bounded by

$$\begin{aligned}&\sum_{j=m-N+1}^{2m-2N-1}\sum_{k=1}^{m-N} f_{2N+2k}u_{2j-2k}f_{4m-2N-2j}\\ &\leqq C_1\sum_{j=m-N+1}^{2m-2N-1}\sum_{k=1}^{m-N}(N+k)^{-3/2}(j-k)^{-3/2}(2m-N-j)^{-3/2},\end{aligned}$$

which is dominated by a constant multiple of

$$N^{-3/2}\sum_{j=m-N+1}^{2m-2N-1}(j-m+N)^{-1/2}(2m-N-j)^{-3/2} \leqq C_2N^{-3}(m-N)^{1/2}.$$

Therefore, if $n = 2m$, we conclude that

$$\begin{aligned}f_{2n} =&u_{2n} - 2\sum_{k=1}^{m} f_{2k}u_{2n-2k} + \sum_{k=1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}u_{2n-2k-2j}\\ &+\sum_{k=N+1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}u_{2n-2k-2j} + O[n^{-5/2}].\end{aligned} \tag{4.14}$$

Since

$$\sum_{k=N+1}^{m}\sum_{j=N+1}^{m} f_{2k}f_{2j}u_{2n-2k-2j} \leqq C_3N^{-3}\sum_{k=N+1}^{m}\sum_{j=N+1}^{m}(2m-k-j)^{-3/2} = O[n^{-5/2}],$$

the forth term of the right hand side of (4.14) is

$$\sum_{k=N+1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j}u_{2n-2k-2j} + O[n^{-5/2}].$$

This immediate implies (4.8).

In the case that $n = 2m+1$, we can apply the same calculation. Detail is left to the reader. We then finish the proof of (4.8).

Before proving Lemma 3.2, we must provide two more lemmatta.

**Lemma 4.2.**

$$f_{2n} = \gamma^2\kappa n^{-3/2} + O[n^{-2}].$$

**Proof.** We first calculate the second term of the right hand side of (4.8). We use $m$ for $[n/2]$ again. By (2.3),

$$\sum_{k=1}^{m} f_{2k}u_{2n-2k} = \kappa\sum_{k=1}^{m} f_{2k}(n-k)^{-3/2} + O\left[\sum_{k=1}^{m} k^{-3/2}(n-k)^{-5/2}\right]. \tag{4.15}$$

It is obvious that the second term of the right hand side of (4.15) is of order $n^{-5/2}$. We estimate the effect of replacing $n-k$ with $n$ in the first term. By the mean value theorem, we have that $0 \leqq (n-k)^{-3/2} - n^{-3/2} \leqq C_4k(n-k)^{-5/2}$ for $1 \leqq k \leqq m$. Then

$$0 \leqq \sum_{k=1}^{m} f_{2k}\left\{(n-k)^{-3/2} - n^{-3/2}\right\} \leqq C_5n^{-5/2}\sum_{k=1}^{m} k^{-1/2} = O[n^{-2}],$$

which yields that the left hand side of (4.15) is equal to

$$\kappa n^{-3/2}\sum_{k=1}^{m} f_{2k} + O[n^{-2}].$$

Remark that

$$\sum_{k=1}^{\alpha} f_{2k} = 1-\gamma-\sum_{k=\alpha+1}^{\infty} f_{2k} = 1-\gamma+O[\alpha^{-1/2}]. \tag{4.16}$$

We therefore obtain that the second term of the right hand side of (4.8) is

$$-2\kappa(1-\gamma)n^{-3/2} + O[n^{-2}].$$

We next calculate the third term of the right hand side of (4.8), which is

$$\kappa \sum_{k=1}^{m} \sum_{j=1}^{m} f_{2k} f_{2j} (n-k-j)^{-3/2} + O\left[ \sum_{k=1}^{m} \sum_{j=1}^{m} k^{-3/2} j^{-3/2} (n-k-j)^{-5/2} \right]. \quad (4.17)$$

Recall that $N$ has been used for $[n/4]$, which is equal to $[m/2]$. We divide the summation in the second term of (4.17) into the following two parts:

$$\sum_{k=1}^{m} \sum_{j=1}^{N} k^{-3/2} j^{-3/2} (n-k-j)^{-5/2}, \quad \sum_{k=1}^{m} \sum_{j=N+1}^{m} k^{-3/2} j^{-3/2} (n-k-j)^{-5/2}.$$

It is easy to see that both summations are of order $n^{-5/2}$. Indeed, the first sum is bounded by a constant multiple of

$$(n-m-N)^{-5/2} \sum_{k=1}^{m} \sum_{j=1}^{N} k^{-3/2} j^{-3/2}$$

and the second one is bounded by a constant multiple of

$$N^{-3/2} \sum_{k=1}^{m} \sum_{j=N+1}^{m} k^{-3/2} (2m-k-j)^{-5/2} \leqq C_6 n^{-3/2} \sum_{k=1}^{m} k^{-3/2} (m-k)^{-3/2}.$$

The estimate of the first term of (4.17) can be obtained in the same manner as the first term of (4.15). We consider the effect of replacing $n-j-k$ with $n$ in the first term of (4.17). Since $(n-k-j)^{-3/2} - n^{-3/2} \leqq C_7(k+j)(n-k-j)^{-5/2}$, it suffices to give an estimate of

$$\sum_{k=1}^{m} \sum_{j=1}^{m} k^{-1/2} j^{-3/2} (n-k-j)^{-5/2}. \quad (4.18)$$

We also divide the summation on $j$ in (4.18) into the case that $1 \leqq j \leqq N$ and the case that $N < j \leqq m$. In the first case, the summation is bounded by a constant multiple of

$$(n-m-N)^{-5/2} \sum_{k=1}^{m} \sum_{j=1}^{N} k^{-1/2} j^{-3/2} = O[n^{-2}].$$

In the second case, the summation is bounded by a constant multiple of

$$N^{-3/2} \sum_{k=1}^{m} \sum_{j=N+1}^{m} k^{-1/2} (2m-k-j)^{-5/2} \leqq C_8 n^{-3/2} \sum_{k=1}^{m} k^{-1/2} (m-k)^{-3/2},$$

which is of order $n^{-2}$. Here we have considered the effect of the summation on $k$ over $1 \leqq k \leqq N$ and that over $N < k \leqq m$. Therefore (4.18) is $O[n^{-2}]$. In virtue of (4.16), this immediately implies that (4.17) is

$$\kappa n^{-3/2} \left( \sum_{k=1}^{m} f_{2k} \right)^2 + O\left[n^{-2}\right] = (1-\gamma)^2 \kappa n^{-3/2} + O[n^{-2}].$$

Applying (2.3) again, we conclude the assertion of this lemma. □

The following lemma is the main tool to calculate the right hand side of (4.8).

**Lemma 4.3.**

$$\sum_{k=1}^{m} f_{2k}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\}=\frac{\gamma^2\kappa}{\sqrt{2}m^2}+O[m^{-5/2}\log m]. \tag{4.19}$$

**Proof.** By Lemma 4.2, the left hand side of (4.19) is equal to

$$\gamma^2\kappa\sum_{k=1}^{m} k^{-3/2}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\} \tag{4.20}$$

$$+O\left[\sum_{k=1}^{m} k^{-2}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\}\right]. \tag{4.21}$$

Since $(2m-k)^{-3/2}-(2m)^{-3/2} \leqq C_9 k(2m-k)^{-5/2}$, the summation on $k$ in (4.21) is dominated by

$$C_9\sum_{k=1}^{m} k^{-1}(2m-k)^{-5/2},$$

which means that (4.21) is of order $m^{-5/2}\log m$. Since $x^3-y^3=(x-y)(x^2+xy+y^2)$, the summation on $k$ in (4.20) is expressed by

$$\sum_{k=1}^{m}\frac{1}{k^{3/2}}\left(\frac{1}{\sqrt{2m-k}}-\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{2m-k}+\frac{1}{\sqrt{2m-k}\sqrt{2m}}+\frac{1}{2m}\right),$$

which is the sum of the following three summations:

$$\frac{1}{\sqrt{2m}}\sum_{k=1}^{m}\frac{1}{\sqrt{k}(\sqrt{2m-k})^3(\sqrt{2m}+\sqrt{2m-k})}, \tag{4.22}$$

$$\frac{1}{2m}\sum_{k=1}^{m}\frac{1}{\sqrt{k}(2m-k)(\sqrt{2m}+\sqrt{2m-k})}, \tag{4.23}$$

$$\frac{1}{(\sqrt{2m})^3}\sum_{k=1}^{m}\frac{1}{\sqrt{k}\sqrt{2m-k}(\sqrt{2m}+\sqrt{2m-k})}. \tag{4.24}$$

Here we have used the following formula:

$$\frac{1}{\sqrt{2m-k}}-\frac{1}{\sqrt{2m}}=\frac{k}{\sqrt{2m-k}\sqrt{2m}}\left(\frac{1}{\sqrt{2m}+\sqrt{2m-k}}\right).$$

By the standard argument of Riemannian integral, we have that (4.22) is

$$\frac{1}{\sqrt{2}m^3}\sum_{k=1}^{m}\frac{1}{\sqrt{k/m}(\sqrt{2-k/m})^3(\sqrt{2}+\sqrt{2-k/m})}$$
$$=\frac{1}{\sqrt{2}m^3}\int_{1/m}^{1}\frac{dx}{\sqrt{x}(\sqrt{2-x})^3(\sqrt{2}+\sqrt{2-x})}+O[m^{-5/2}].$$

Substituting $y = \sqrt{x/2}$ in the integral on $x$, we obtain that (4.22) is

$$\frac{1}{2m^2}\int_{1/\sqrt{2m}}^{1/\sqrt{2}} \frac{dy}{(\sqrt{1-y^2})^3(1+\sqrt{1-y^2})} + O[m^{-5/2}].$$

We can calculate (4.23) and (4.24) in an analogous manner, and conclude that (4.23) is

$$\frac{1}{2m^2}\int_{1/\sqrt{2m}}^{1/\sqrt{2}} \frac{dy}{(1-y^2)(1+\sqrt{1-y^2})} + O[m^{-5/2}]$$

and that (4.24) is

$$\frac{1}{2m^2}\int_{1/\sqrt{2m}}^{1/\sqrt{2}} \frac{dy}{\sqrt{1-y^2}(1+\sqrt{1-y^2})} + O[m^{-5/2}].$$

Therefore the leading term of (4.20) is equal to

$$\frac{\gamma^2\kappa}{2m^2}\int_{1/\sqrt{2m}}^{1/\sqrt{2}} \left\{\frac{1}{(\sqrt{1-y^2})^3} + \frac{1}{1-y^2} + \frac{1}{\sqrt{1-y^2}}\right\}\frac{1}{1+\sqrt{1-y^2}}dy,$$

which coincides with

$$\frac{\gamma^2\kappa}{2m^2}\int_{1/\sqrt{2m}}^{1/\sqrt{2}} \frac{1}{y^2}\left\{\frac{1}{(\sqrt{1-y^2})^3} - 1\right\}dy \tag{4.25}$$

since $x^{-3}+x^{-2}+x^{-1} = (x^{-3}-1)/(1-x)$ and the remaining term of (4.20) is of order $m^{-5/2}$. Moreover the fact that

$$\frac{d}{dy}\left(-\frac{1}{y\sqrt{1-y^2}} + \frac{2y}{\sqrt{1-y^2}} + \frac{1}{y}\right) = \frac{1}{y^2}\left\{\frac{1}{(\sqrt{1-y^2})^3} - 1\right\}$$

implies that (4.25) is $\gamma^2\kappa/\sqrt{2}m^2 + O[m^{-5/2}]$, which yields (4.19). □

We are ready to show Lemma 3.2. We consider only the case that $n = 2m$ since we can show the lemma in the case that $n = 2m+1$ analogously to this case. Lemma 4.3 and (4.15) yields that the second term of the right hand side of (4.8) is

$$\kappa\sum_{k=1}^{m} f_{2k}(2m)^{-3/2} + \frac{\gamma^2\kappa^2}{\sqrt{2}m^2} + O[m^{-5/2}\log m].$$

Lemma 4.2 immediately implies that

$$\sum_{k=1}^{m} f_{2k} = 1 - \gamma - 2\gamma^2\kappa m^{-1/2} + O[m^{-1}]. \tag{4.26}$$

Then

$$\sum_{k=1}^{m} f_{2k}u_{4m-2k} = (1-\gamma)\kappa(2m)^{-3/2} + O[m^{-5/2}\log m].$$

We have already see that the third term of the right hand side of (4.8) is

$$\kappa\sum_{k=1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j}(2m-k-j)^{-3/2}+O[m^{-5/2}] \tag{4.27}$$

in the proof of Lemma 4.2. We recall that $N=[n/4]=[m/2]$ and write $M$ for $[m^{3/4}]$. The double sum in (4.27) is divided into the following three parts:

$$\sum_{k=1}^{M}\sum_{j=1}^{m} f_{2k}f_{2j}(2m-k-j)^{-3/2}, \tag{4.28}$$

$$\sum_{k=M+1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}(2m-k-j)^{-3/2}, \tag{4.29}$$

$$\sum_{k=M+1}^{m}\sum_{j=N+1}^{m} f_{2k}f_{2j}(2m-k-j)^{-3/2}. \tag{4.30}$$

It is easy to see that (4.30) is of order $m^{-17/8}$. Indeed, by (2.3), it is dominated by a constant multiple of

$$m^{-21/8}\sum_{k=M+1}^{m}\sum_{j=N+1}^{m}(2m-k-j)^{-3/2}.$$

We first estimate the effect of replacing $2m-k-j$ with $2m-j$ in (4.28). By the mean value theorem,

$$\begin{aligned}&\sum_{k=1}^{M}\sum_{j=1}^{m} f_{2k}f_{2j}\left\{(2m-k-j)^{-3/2}-(2m-j)^{-3/2}\right\}\\ &\leqq C_{10}\sum_{k=1}^{M}\sum_{j=1}^{m} k^{-1/2}j^{-3/2}(2m-k-j)^{-5/2},\end{aligned}$$

which is bounded by

$$C_{10}(m-M)^{-5/2}\sum_{k=1}^{M}\sum_{j=1}^{m} k^{-1/2}j^{-3/2}=O[m^{-17/8}].$$

This means that (4.28) is

$$\sum_{k=1}^{M}\sum_{j=1}^{m} f_{2k}f_{2j}(2m-j)^{-3/2}+O[m^{-17/8}].$$

We next estimate the effect of replacing $2m-j$ with $2m$ in this double sum. It follows from (4.16) and (4.19) that

$$\sum_{k=1}^{M}\sum_{j=1}^{m} f_{2k}f_{2j}\left\{(2m-j)^{-3/2}-(2m)^{-3/2}\right\} = \frac{\gamma^2\kappa}{\sqrt{2}m^2}\sum_{k=1}^{M} f_{2k} + O[m^{-5/2}\log m]$$
$$= \frac{\gamma^2(1-\gamma)\kappa}{\sqrt{2}m^2} + O[m^{-19/8}],$$

which yields that (4.28) is equal to

$$(2m)^{-3/2}\sum_{k=1}^{M}\sum_{j=1}^{m} f_{2k}f_{2j} + \frac{\gamma^2(1-\gamma)\kappa}{\sqrt{2}m^2} + O[m^{-17/8}].$$

The effect of replacing $2m-k-j$ with $2m-k$ in (4.29) is of order

$$\sum_{k=M+1}^{m}\sum_{j=1}^{N} k^{-3/2}j^{-1/2}(2m-k-j)^{-5/2},$$

which is bounded by a constant multiple of

$$M^{-3/2}\sum_{j=1}^{N} j^{-1/2}(m-j)^{-3/2} = O[m^{-17/8}].$$

Therefore the leading term of (4.29) is

$$\sum_{k=M+1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}(2m-k)^{-3/2} \tag{4.31}$$

and the remaining term of (4.29) is of order $m^{-17/8}$. We divide (4.31) into the following two parts:

$$\sum_{k=M+1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}(2m)^{-3/2}, \tag{4.32}$$

$$\sum_{k=M+1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\}. \tag{4.33}$$

It follows that

$$\sum_{k=M+1}^{m}\sum_{j=N+1}^{m} f_{2k}f_{2j}(2m)^{-3/2} = O[m^{-17/8}],$$

which yields that (4.32) is

$$(2m)^{-3/2}\sum_{k=M+1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j} + O[m^{-17/8}].$$

Moreover we have that

$$\sum_{k=1}^{M}\sum_{j=1}^{N} f_{2k}f_{2j}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\} \leqq C_{11}\sum_{k=1}^{M} k^{-1/2}(2m-k)^{-5/2},$$

which is of order $m^{-17/8}$. Therefore (4.33) is

$$\sum_{k=1}^{m}\sum_{j=1}^{N} f_{2k}f_{2j}\left\{(2m-k)^{-3/2}-(2m)^{-3/2}\right\} + O[m^{-17/8}],$$

of which the first term is equal to $\gamma^2(1-\gamma)\kappa/\sqrt{2}m^2$. Here we have applied (4.16) and (4.19). We hence obtain that (4.31) and also (4.29) are

$$(2m)^{-3/2}\sum_{k=M+1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j} + \frac{\gamma^2(1-\gamma)\kappa}{\sqrt{2}m^2} + O[m^{-17/8}].$$

Consequently we have that

$$\begin{aligned}\sum_{k=1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j}u_{4m-2k-2j} =&\kappa(2m)^{-3/2}\sum_{k=1}^{m}\sum_{j=1}^{m} f_{2k}f_{2j} \\ &+ \frac{\sqrt{2}\gamma^2(1-\gamma)\kappa^2}{m^2} + O[m^{-17/8}].\end{aligned} \tag{4.34}$$

By (4.26), the first term of the right hand side of (4.34) is

$$(1-\gamma)^2\kappa(2m)^{-3/2} - \frac{\sqrt{2}\gamma^2(1-\gamma)\kappa^2}{m^2} + O[m^{-5/2}],$$

which means that the left hand side of (4.34) is

$$(1-\gamma)^2\kappa(2m)^{-3/2} + O[m^{-17/8}].$$

Then (4.8) yields

$$f_{4m} = \gamma^2\kappa(2m)^{-3/2} + O[m^{-17/8}],$$

which is equivalent to the assertion of Lemma 3.2 if $n = 2m$. This complete the proof of Lemma 3.2.

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Yuji Hamana
Department of Mathematics
Kumamoto University
Kumamoto 860-8555, Japan
e-mail: hamana@math.sci.kumamoto-u.ac.jp