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On the expected volume of the Wiener sausage

Dedicated to Professor Yasunari Higuchi on his 60th birthday

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Abstract. We consider the expected volume of the Wiener sausage on the time interval $[0,t]$ associated with a closed ball. Let $L(t)$ be the expected volume minus the volume of the ball. We obtain that $L(t)$ is asymptotically equal to a constant multiple of $t^{1/2}$ as t tends to 0 and that it is represented as an absolutely convergent power series of $t^{1/2}$ for any $t > 0$ in the odd dimensional cases. Moreover, the explicit form of $L(t)$ can be given in five and seven dimensional cases.

1. Introduction.

In connection with heat conduction problems, the volume of the Wiener sausage up to time t associated with a non-polar compact set has been investigated for a long time. The expected volume of the Wiener sausage is interpreted as the total energy flow from the non-polar set. In the two dimensional case, Spitzer [**14**] showed that the leading term is $2\pi t/\log t$ for large t. It is remarkable that the leading term is independent of the non-polar set. In three or more dimensional cases, the expected volume of the Wiener sausage is asymptotically equal to t multiple of the capacity of the non-polar compact set as $t \to \infty$, which can be found in Getoor [**4**] and Spitzer [**14**]. In addition, Le Gall [**10**] improved these results and provided some lower terms.

Several results on limit theorems for the Wiener sausage have been established. The law of large numbers was proved by Whitman in three or more dimensional cases which is described in Itô-McKean [**7**] and by Le Gall [**8**] in the two dimensional case. Le Gall [**9**] also established the central limit theorem. The results concerning large deviations are given in van den Berg-Bolthausen-den Hollander [**2**] and Hamana-Kesten [**5**].

In this article we will treat the Wiener sausage associated with a closed ball.

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For $t > 0$ let $L(t)$ be the expected volume of the Wiener sausage up to time t from which the closed ball is removed. Asymptotic behavior of $L(t)$ as $t \to 0$ can be obtained by its Laplace transform and the Tauberian theorem. We can find that the leading term of $L(t)$ is a constant multiple of $t^{1/2}$ and that the surface area of the unit ball appears in the constant, which are described in Section 3.

On and after Section 4, we deal with odd dimensional cases. In the one dimensional case, Berezhkovskii, Makhnovskii and Suris have investigated in [**1**]. In addition, we can easily calculate $L(t)$ in this case. If the dimension is three, Spitzer [**14**] has already given the form of $L(t)$ explicitly. The main result in this paper is that $L(t)$ can be represented as an absolutely convergent power series of $t^{1/2}$ for any $t > 0$ in higher dimensional cases. In five and seven dimensional cases in particular, we can give the explicit form of $L(t)$. Section 6 is devoted to them.

2. Notation and preliminaries.

Let $r > 0$ be a fixed number. We use the notation D for $\{x \in \boldsymbol{R}^d; \|x\| \leqq r\}$, where $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$ for $x = (x_1, \ldots, x_d) \in \boldsymbol{R}^d$. A Brownian motion on $\boldsymbol{R}^d$ will be denoted by $\{B(t)\}_{t \geqq 0}$. For $t \geqq 0$ let

$$C(t) = \bigl\{x \in \boldsymbol{R}^d; x + B(s) \in D \text{ for some } s \in [0, t]\bigr\},$$

which is called the Wiener sausage for $\{B(t)\}_{t \geqq 0}$ associated with the set D on the time interval $[0, t]$. A simple calculation shows that the expected volume of $C(t)$ is given by

$$\int_{\boldsymbol{R}^d} P_x(\tau \leqq t) dx,$$

where τ is the first hitting time to D of $\{B(t)\}_{t \geqq 0}$ and P_x is the probability measure of events related to the Brownian motion starting from $x \in \boldsymbol{R}^d$. The notation $|A|$ will be used to denote the volume of a subset A in $\boldsymbol{R}^d$, and then we have that

$$E|C(t)| = |D| + \int_{\boldsymbol{R}^d \setminus D} P_x(\tau \leqq t) dx. \tag{2.1}$$

According to the result in Hunt [**6**], $P_x(\tau \leqq t)$ is the unique solution of the heat conduction problem

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x)$$

for $t > 0$ and $x \in \boldsymbol{R}^d \setminus D$ subject to the initial condition $u(0, x) = 0$ for $x \in \boldsymbol{R}^d \setminus D$ and the boundary condition $u(t, y) = 1$ for $t > 0$ and $y \in D$. Hence $P_x(\tau \leqq t)$ may be interpreted as the temperature at time t at the point $x \in \boldsymbol{R}^d$. We write $L(t)$ for the second term of the right hand side of (2.1). Then $L(t)$ implies the total energy flow in time t from the set D into the surrounding medium $\boldsymbol{R}^d \setminus D$.

In the one dimensional case, it is easy to see that

$$L(t) = 2\left(\frac{2}{\pi}\right)^{1/2} t^{1/2}$$

with the help of the fact that

$$P_x(\tau \leqq t) = \int_0^t \frac{|x| - r}{\sqrt{2\pi s^3}} \exp\left[-\frac{(|x| - r)^2}{2s}\right] ds$$

for $|x| > r$, which is given in Itô-McKean [**7**, p. 25]. We remark that $L(t)$ is independent of r. In the three dimensional case, Spitzer [**14**] showed that

$$L(t) = 2\pi r t + 4(2\pi)^{1/2} r^2 t^{1/2}.$$

This equality can be also obtained directly by the following well-known formula:

$$P_x(\tau \leqq t) = \frac{r(\|x\| - r)}{\|x\|} \int_0^t \frac{1}{\sqrt{2\pi s^3}} \exp\left[-\frac{(\|x\| - r)^2}{2s}\right] ds$$

for $\|x\| > r$. (See Le Gall [**10**].) In higher dimensional cases, although there is not such a useful explicit formula, the Laplace transform of τ can be computed. For $x \in \boldsymbol{R}^d$ let μ_x be the probability distribution of τ under P_x. Since μ_x is equal to the distribution of the first hitting time to r of the d dimensional Bessel process starting from $\|x\|$, we have that

$$\int_{\boldsymbol{R}_+} e^{-\lambda t} d\mu_x(t) = \frac{\|x\|^{-\nu} K_\nu(\|x\|\sqrt{2\lambda})}{r^{-\nu} K_\nu(r\sqrt{2\lambda})} \tag{2.2}$$

for $\lambda > 0$ and $\|x\| \geqq r$, where $\boldsymbol{R}_+ = \{t \in \boldsymbol{R}; t \geqq 0\}$, $\nu = d/2 - 1$ and K_μ is the modified Bessel function of the second kind of order μ. This formula can be found in Borodin-Salminen [**3**, p. 387] and Itô-McKean [**7**, p. 129]. We note that K_μ is also called the Macdonald function of order μ.

In general, K_μ is the function defined on $\boldsymbol{C}$ for each complex number μ. In this paper, however, we need the value of $K_\mu(x)$ only in the case that x and μ are real satisfying with $x > 0$. The remainder of this section is devoted to some properties of modified Bessel functions, which are given in Magnus-Oberhettinger-Soni [**11**]. Since K_μ coincides with $K_{-\mu}$ for $\mu \in \boldsymbol{R}$, it is enough to consider the case that $\mu \geqq 0$. For each $\mu \geqq 0$ and integer $M \geqq 1$ we have the following asymptotic expansion of Hankel type:

$$K_\mu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[\sum_{m=0}^{M-1} \frac{(\mu + 1/2)_m (\mu - m + 1/2)_m}{(2x)^m m!} + O(x^{-M}) \right] \tag{2.3}$$

as x tends to ∞, where

$$(a)_m = \begin{cases} a(a+1)\cdots(a+m-1) & \text{if } m \geqq 1, \\ 1 & \text{if } m = 0 \end{cases}$$

for $a \in \boldsymbol{R}$. This immediately yields that

$$\lim_{x \to \infty} x^\mu K_{\mu'}(x) = 0, \tag{2.4}$$

$$\lim_{x \to \infty} \frac{K_\mu(x)}{K_{\mu'}(x)} = 1 \tag{2.5}$$

for μ, $\mu' \geqq 0$. The following formula with respect to indefinite integrals is quite useful for calculating the Laplace transform of L:

$$\int x^{\mu+1} K_\mu(x) dx = -x^{\mu+1} K_{\mu+1}(x) + C, \tag{2.6}$$

where C is an arbitrary constant. In the case that μ is a half integer, we can express K_μ explicitly and obtain that

$$K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left[\sum_{m=0}^{n} \frac{\langle n, m \rangle}{(2x)^m} \right] \tag{2.7}$$

for each integer $n \geqq 0$. For simplicity, we have used the following notation:

$$\langle n, m \rangle = \begin{cases} \dfrac{(n+m)!}{m!(n-m)!} & \text{if } n \geqq m, \\ 0 & \text{if } n < m. \end{cases}$$

Throughout this paper, for a suitable function f, the notation $\mathscr{L}[f]$ implies the Laplace transform of f and the inverse Laplace transform of f is denoted by $\mathscr{L}^{-1}[f]$.

3. Asymptotic behavior of $L(t)$ as t is small.

Applying the Tauberian theorem, we can obtain asymptotic behavior of $L(t)$ as t tends to 0 if the behavior of $\mathscr{L}[L](\lambda)$ for large λ is provided. In fact, we can supply the explicit form of $\mathscr{L}[L](\lambda)$ for $\lambda > 0$.

PROPOSITION 3.1. *We have that*

$$\mathscr{L}[L](\lambda) = \frac{c_d}{\lambda^{3/2}} \frac{K_{\nu+1}(r\sqrt{2\lambda})}{K_\nu(r\sqrt{2\lambda})} \tag{3.1}$$

for $\lambda > 0$, *where* $c_d = S_{d-1} r^{d-1}/\sqrt{2}$ *and* S_{d-1} *is the surface area of* $d-1$ *dimensional unit sphere.*

PROOF. For $s, t \in \boldsymbol{R}$ let $\chi(s,t)$ be 1 if $s \leqq t$ and 0 otherwise. Then we have that

$$\begin{aligned}\mathscr{L}[L](\lambda) &= \int_{\boldsymbol{R}_+} e^{-\lambda t}\left[\int_{\boldsymbol{R}^d\setminus D}\left[\int_{[0,t]} d\mu_x(s)\right]dx\right]dt \\ &= \int_{\boldsymbol{R}_+}\left[e^{-\lambda t}\int_{\boldsymbol{R}^d\setminus D}\left[\int_{\boldsymbol{R}_+}\chi(s,t)d\mu_x(s)\right]dx\right]dt\end{aligned}$$

for $\lambda > 0$. By the Fubini theorem, the last triple integral is equal to

$$\int_{\boldsymbol{R}^d\setminus D}\left[\int_{\boldsymbol{R}_+}\left[\int_{\boldsymbol{R}_+} e^{-\lambda t}\chi(s,t)dt\right]d\mu_x(s)\right]dx = \int_{\boldsymbol{R}^d\setminus D}\left[\frac{1}{\lambda}\int_{\boldsymbol{R}_+} e^{-\lambda s}d\mu_x(s)\right]dx.$$

Therefore, by (2.2) and the change of variables formula,

$$\begin{aligned}\mathscr{L}[L](\lambda) &= \frac{1}{\lambda}\int_{\boldsymbol{R}^d\setminus D}\frac{\|x\|^{-\nu}K_\nu(\|x\|\sqrt{2\lambda})}{r^{-\nu}K_\nu(r\sqrt{2\lambda})}dx \\ &= \frac{S_{d-1}}{\lambda r^{-\nu}K_\nu(r\sqrt{2\lambda})}\int_r^\infty \rho^{d-\nu-1}K_\nu(\rho\sqrt{2\lambda})d\rho,\end{aligned}$$

which is equal to

$$\frac{S_{d-1}}{\lambda r^{-\nu} K_\nu(r\sqrt{2\lambda})\sqrt{2\lambda}^{d-\nu}} \int_{r\sqrt{2\lambda}}^{\infty} y^{d-\nu-1} K_\nu(y)dy. \tag{3.2}$$

Recall that $\nu = d/2-1$, and then it follows that $d-\nu-1 = \nu+1$. Applying (2.4) and (2.6) to the integral in (3.2), we obtain that

$$\int_{r\sqrt{2\lambda}}^{\infty} y^{d-\nu-1} K_\nu(y)dy = \big(r\sqrt{2\lambda}\big)^{\nu+1} K_{\nu+1}\big(r\sqrt{2\lambda}\big),$$

which yields (3.1). □

Applying (2.5) to (3.1),

$$\mathscr{L}[L](\lambda) = \frac{c_d}{\lambda^{3/2}} + o\left(\frac{1}{\lambda^{3/2}}\right)$$

as $\lambda \to \infty$. The Tauberian theorem yields that

$$L(t) = \frac{2c_d}{\sqrt{\pi}} t^{1/2} + o(t^{1/2})$$

as $t \to 0$. This means that we have finished to give a proof of the following.

Corollary 3.2. *We have that*

$$L(t) = \sqrt{\frac{2}{\pi}} S_{d-1} r^{d-1} t^{1/2} + o(t^{1/2}) \tag{3.3}$$

as $t \to 0$.

The first term of the right hand side of (3.3) is the brief energy flow from a ball of which the temperature is kept at one when the ball is thrown into the medium with temperature zero.

By (2.3), Proposition 3.1 also yields that $\mathscr{L}[L](\lambda)$ can be represented as a power series of $\lambda^{-1/2}$ if λ is sufficiently large. Hence we would like to expect that $L(t)$ can be represented as a power series of $t^{1/2}$ if t is sufficiently small. However this argument is not correct in general. In virtue of the fact that K_μ has the explicit form like (2.7) if μ is a half integer, $\mathscr{L}[L](\lambda)$ can be represented as a rational function of $\lambda^{1/2}$. This implies that the inverse Laplace transform of $\mathscr{L}[L]$ can be calculated in principle. In the next section, we will supply a representation of $L(t)$.

4. Power series representation of $L(t)$.

In this section we will consider odd dimensional cases. Since the explicit form of $L(t)$ has been obtained in the one and three dimensional cases, it is sufficient to consider higher dimensional cases. The purpose of this section is to establish the following theorem.

THEOREM 4.1. *If d is odd and more than or equal to five, we have that*

$$L(t) = c_d \sum_{n=1}^{\infty} \alpha_n^{(d)} t^{n/2} \tag{4.1}$$

for any $t > 0$ and that the right hand side of (4.1) converges absolutely, where $\{\alpha_n^{(d)}\}_{n=1}^{\infty}$ is the sequence of real numbers defined by

$$\alpha_n^{(d)} = \frac{\beta_{n-1}^{(d)}}{(\sqrt{2}r)^{n-1}\Gamma(n/2+1)}. \tag{4.2}$$

Here Γ is the gamma function and the sequence $\{\beta_n^{(d)}\}_{n=0}^{\infty}$ is determined by

$$\frac{1}{2^k}\left\langle \frac{d-1}{2}, k \right\rangle = \sum_{j=0}^{k} \frac{1}{2^{k-j}} \left\langle \frac{d-3}{2}, k-j \right\rangle \beta_j^{(d)} \tag{4.3}$$

for $k \geqq 0$.

REMARK. It follows from (4.3) that

$$\beta_0^{(d)} = 1, \quad \beta_1^{(d)} = \frac{d-1}{2}, \quad \beta_2^{(d)} = \frac{(d-1)(d-3)}{8}, \quad \beta_3^{(d)} = -\frac{(d-1)(d-3)}{8}.$$

Moreover, in general, we obtain that

$$\beta_n^{(d)} = \frac{1}{2^n} \sum_{j=1}^{n} \sum_{m=1}^{j} (-1)^m \sum_{\substack{k_1+\cdots+k_m=j \\ k_1 \geqq 1, \ldots, k_m \geqq 1}} \left\langle \frac{d-3}{2}, k_1 \right\rangle \cdots \left\langle \frac{d-3}{2}, k_m \right\rangle \left\langle \frac{d-1}{2}, n-j \right\rangle + \frac{1}{2^n} \left\langle \frac{d-1}{2}, n \right\rangle$$

for $n \geqq 1$.

Before proving Theorem 4.1, we consider $L(t)$ in the five and seven dimensional cases. It is not difficult to see from (4.3) that $\beta_0^{(5)} = 1$, $\beta_1^{(5)} = 2$, $\beta_n^{(5)} = (-1)^n$ for $n \geqq 2$ and that $\beta_0^{(7)} = 1$, $\beta_1^{(7)} = \beta_2^{(7)} = 3$, $\beta_3^{(7)} = -3$ and each $\beta_n^{(7)}$ for $n \geqq 4$ is determined by $\beta_{k+2}^{(7)} = -3(\beta_{k+1}^{(7)} + \beta_k^{(7)})$, that is,

$$\beta_n^{(7)} = -\frac{1}{2}\left[\left(\frac{-3-\sqrt{3}\,i}{2}\right)^{n-1} + \left(\frac{-3+\sqrt{3}\,i}{2}\right)^{n-1}\right],$$

where i denotes the imaginary unit. By Theorem 4.1, these yield that if $d = 5$,

$$L(t) = S_4\left[\sqrt{\frac{2}{\pi}}r^4t^{1/2} + r^3t + \frac{1}{3}\sqrt{\frac{2}{\pi}}r^2t^{3/2} - \frac{1}{8}rt^2 + \frac{1}{15}\sqrt{\frac{2}{\pi}}t^{5/2} - \cdots\right]$$

and that if $d = 7$,

$$L(t) = S_6\left[\sqrt{\frac{2}{\pi}}r^6t^{1/2} + \frac{3}{2}r^5t + \sqrt{\frac{2}{\pi}}r^4t^{3/2} + \frac{3}{8}r^3t^2 + \frac{3}{16}rt^3 - \cdots\right].$$

We note that the right hand side does not have the term of $t^{5/2}$ if $d = 7$. In addition, we can obtain the explicit form of $L(t)$ in the five and seven dimensional cases. They will be discussed in Section 6.

The remainder of this section is devoted to the proof of Theorem 4.1. We first supply one lemma concerning the inverse Laplace transform.

Lemma 4.2. *Let f be a function whose Laplace transform exists. Then the inverse Laplace transform of $\lambda^{-3/2}\mathscr{L}[f](\lambda^{1/2})$ is equal to*

$$\int_0^t \left[\frac{1}{\sqrt{\pi w}}\int_0^\infty e^{-u^2/4w}f(u)du\right]dw. \tag{4.4}$$

Proof. We can prove (4.4) by combining three formulae with respect to inverse Laplace transforms, which we can find in Prudnikov-Brychkov-Marichev [**13**].

The first one is that the inverse transform of $\lambda^{-1/2}\mathscr{L}[f](\lambda^{1/2})$ is

$$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}f(u)du.$$

The second one is that the inverse transform of λ^{-1} is 1, and the last one is that the inverse transform of $\mathscr{L}[f] \cdot \mathscr{L}[g]$ is

$$\int_0^t f(t-w)g(w)dw.$$

Thus the inverse transform of $\lambda^{-1} \cdot \lambda^{-1/2}\mathscr{L}[f](\lambda^{1/2})$ is equal to (4.4). $\square$

We are ready to prove Theorem 4.1. Let $d = 2\varrho + 3$ for some positive integer ϱ. Recall that $\nu = d/2 - 1$ again, and then $\nu = \varrho + 1/2$. It turns that Proposition 3.1 implies that

$$\mathscr{L}[L](\lambda) = \frac{c_d}{\lambda^{3/2}} \frac{K_{\varrho+3/2}(r\sqrt{2\lambda})}{K_{\varrho+1/2}(r\sqrt{2\lambda})}. \tag{4.5}$$

By a simple calculation, the equality (2.7) yields that

$$\mathscr{L}[L](\lambda) = \frac{c_d}{\lambda^{3/2}} \left[1 + \frac{f_\varrho(r\sqrt{2\lambda})}{g_\varrho(r\sqrt{2\lambda})}\right]$$

for some suitable polynomials f_ϱ and g_ϱ of degree ϱ and $\varrho + 1$ respectively. Therefore we obtain that

$$\mathscr{L}[L](\lambda) = \frac{c_d}{\lambda^{3/2}} + \frac{\bar{c}_d}{(2r^2\lambda)^{3/2}} \frac{f_\varrho(\sqrt{2r^2\lambda})}{g_\varrho(\sqrt{2r^2\lambda})}, \tag{4.6}$$

where $\bar{c}_d = 2^{3/2} r^3 c_d$. The inverse Laplace transform of the first term of the right hand side of (4.6) is a constant multiple of $t^{1/2}$. Therefore we now concentrate on considering the remaining term.

Let h_ϱ be the inverse Laplace transforms of the second term of the right hand side of (4.6). Then Lemma 4.2 shows that

$$h_\varrho(t) = \frac{\bar{c}_d}{2r^2} \int_0^{t/2r^2} \frac{1}{\sqrt{\pi w}} \left[\int_0^\infty e^{-u^2/4w} \mathscr{L}^{-1}\left[\frac{f_\varrho}{g_\varrho}\right](u)du\right] dw.$$

The partial fraction decomposition yields that $f_\varrho(\xi)/g_\varrho(\xi)$ is the linear combination of rational functions of the following three types:

$$p_N^a(\xi) = \frac{1}{(\xi - a)^N}, \quad q_N^{a,b}(\xi) = \frac{1}{\{(\xi-a)^2 + b^2\}^N}, \quad r_N^{a,b}(\xi) = \frac{\xi - a}{\{(\xi-a)^2 + b^2\}^N},$$

where $a, b \in \boldsymbol{R}$ with $b > 0$ and N is positive integer. We note that each denominator in $p_N^a(\xi)$, $q_N^{a,b}(\xi)$ and $r_N^{a,b}(\xi)$ is the factor of $g_\varrho(\xi)$. Then $h_\varrho(t)$ is the linear combination of $P_N^a(t/2r^2)$, $Q_N^{a,b}(t/2r^2)$, $R_N^{a,b}(t/2r^2)$, where

$$P_N^a(t) = \int_0^t \frac{1}{\sqrt{w}} \left[\int_0^\infty e^{-u^2/4w} \mathscr{L}^{-1}[p_N^a](u) du \right] dw,$$

$$Q_N^{a,b}(t) = \int_0^t \frac{1}{\sqrt{w}} \left[\int_0^\infty e^{-u^2/4w} \mathscr{L}^{-1}[q_N^{a,b}](u) du \right] dw,$$

$$R_N^{a,b}(t) = \int_0^t \frac{1}{\sqrt{w}} \left[\int_0^\infty e^{-u^2/4w} \mathscr{L}^{-1}[r_N^{a,b}](u) du \right] dw.$$

It is easy to see that

$$\mathscr{L}^{-1}[p_N^a](t) = \frac{e^{at} t^{N-1}}{\Gamma(N)}.$$

For $\alpha, t \in \boldsymbol{R}$ with $t > 0$ and an integer $m \geqq 0$ let

$$\gamma_m^\alpha(t) = \int_0^t \frac{1}{\sqrt{w}} \left[\int_0^\infty e^{-u^2/4w + \alpha u} u^m du \right] dw.$$

Then $P_N^a(t) = \gamma_{N-1}^a(t)/\Gamma(N)$. Moreover, the representations of $Q_N^{a,b}$ and $R_N^{a,b}$ can be found as follows.

Lemma 4.3. *We have that*

$$Q_N^{a,b}(t) = P_{2N}^a(t) + \frac{1}{2^{N-1}\Gamma(N)} \sum_{n=1}^\infty \frac{(-1)^n b^{2n}}{(2n)!} \frac{(2n-1)!!}{(2n+2N-1)!!} \gamma_{2N+2n-1}^a(t), \quad (4.7)$$

$$R_N^{a,b}(t) = P_{2N-1}^a(t) + \frac{1}{2^{N-1}\Gamma(N)} \sum_{n=1}^\infty \frac{(-1)^n b^{2n}}{(2n)!} \frac{(2n-1)!!}{(2n+2N-3)!!} \gamma_{2N+2n-2}^a(t). \quad (4.8)$$

The proof of this lemma is postponed to Section 5. Lemma 4.3 implies that it is sufficient to show that $\gamma_m^\alpha(t)$ is represented as a power series of $\sqrt{t}$ for the establishment of (4.1). Changing variables of the double integral on u and w in $\gamma_m^\alpha(t)$ by $x = u/2\sqrt{w}$ and $y = \sqrt{w}$,

$$\gamma_m^\alpha(t) = 2^{m+2}\int_0^{\sqrt{t}}\left[\int_0^\infty e^{-x^2+2\alpha xy}x^m y^{m+1}dx\right]dy$$
$$= 2^{m+2}\int_0^{\sqrt{t}}\left[\int_0^\infty e^{-x^2}x^m y^{m+1}\sum_{n=0}^\infty \frac{(2\alpha xy)^n}{n!}dx\right]dy.$$

Here 0^0 has been interpreted as 1. To change the order of the last double integral and the summation, we need to show that

$$\sum_{n=0}^\infty \int_0^{\sqrt{t}}\left[\int_0^\infty e^{-x^2}x^m y^{m+1}\frac{(2|\alpha|xy)^n}{n!}dx\right]dy \tag{4.9}$$

converges. It is easy to obtain the convergence. Indeed, the monotone convergence theorem yields that (4.9) is equal to

$$\int_0^{\sqrt{t}} y^{m+1}\left[\int_0^\infty e^{-x^2+2|\alpha|xy}x^m dx\right]dy,$$

which is not larger than

$$\sqrt{t}^{m+2}\int_0^\infty e^{-x^2+2|\alpha|\sqrt{t}x}x^m dx.$$

Since this improper integral converges, we can conclude the convergence of (4.9). Therefore, by the Fubini theorem,

$$\gamma_m^\alpha(t) = 2^{m+2}\sum_{n=0}^\infty \frac{(2\alpha)^n}{n!}\frac{\sqrt{t}^{m+n+2}}{m+n+2}\int_0^\infty e^{-x^2}x^{m+n}dx. \tag{4.10}$$

By Lemma 4.3 and (4.10), we consequently obtain that each $P_N^a(t)$, $Q_N^{a,b}(t)$ and $R_N^{a,b}(t)$ is represented as a power series of $t^{1/2}$. In order to conclude (4.1), we need to show that these series converge absolutely for any $t > 0$. Indeed, it can be proved in the following way.

For an integer $m \geqq 1$ and real numbers $\alpha \geqq 0$, $\beta > 0$, $t > 0$ let

$$\xi_m^\alpha(t) = \sum_{n=0}^\infty \frac{(2\alpha)^n}{n!}\frac{\sqrt{t}^{m+n+1}}{m+n+1}\int_0^\infty e^{-x^2}x^{m+n-1}dx,$$

$$\eta_m^{\alpha,\beta}(t) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\beta^{2n}}{(2n)!} \frac{(2n-1)!!}{(2n+2m-1)!!} 2^{2m+2n+1} \frac{(2\alpha)^k}{k!} \times \frac{\sqrt{t}^{2m+2n+k+1}}{2m+2n+k+1} \int_0^{\infty} e^{-x^2} x^{2m+2n+k-1} dx,$$

$$\zeta_m^{\alpha,\beta}(t) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\beta^{2n}}{(2n)!} \frac{(2n-1)!!}{(2n+2m-3)!!} 2^{2m+2n} \frac{(2\alpha)^k}{k!} \times \frac{\sqrt{t}^{2m+2n+k}}{2m+2n+k} \int_0^{\infty} e^{-x^2} x^{2m+2n+k-2} dx.$$

It is sufficient to show that

$$\xi_m^{\alpha}(t) \leqq \begin{cases} K_1 t^{m+1/2} e^{4\alpha^2 t} & \text{if } \alpha \neq 0 \text{ and } t \geqq 1/4\alpha^2, \\ K_2 & \text{if } \alpha \neq 0 \text{ and } t < 1/4\alpha^2, \\ K_3 t^{(m+1)/2} & \text{if } \alpha = 0, \end{cases} \tag{4.11}$$

$$\eta_m^{\alpha,\beta}(t) \leqq \begin{cases} K_4 t^{2m+1/2} e^{4(\alpha+\beta)^2 t} & \text{if } t \geqq 1/4(\alpha+\beta)^2, \\ K_5 & \text{if } t < 1/4(\alpha+\beta)^2, \end{cases} \tag{4.12}$$

$$\zeta_m^{\alpha,\beta}(t) \leqq \begin{cases} K_6 t^{2m-1/2} e^{4(\alpha+\beta)^2 t} & \text{if } t \geqq 1/4(\alpha+\beta)^2, \\ K_7 & \text{if } t < 1/4(\alpha+\beta)^2. \end{cases} \tag{4.13}$$

Here K_1, K_2, K_3, K_4, K_5, K_6 and K_7 are some suitable constants, which are all independent of t. In order to see them, the following lemma is quite useful.

Lemma 4.4. *Let $m \geqq 0$ be a given integer. For a real number $\alpha > 0$ there exist constants A_1 and A_2, which are independent of α, such that*

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^{\infty} e^{-x^2} x^{m+n} dx \leqq \begin{cases} A_1 \alpha^{m+1} e^{\alpha^2} & \text{if } \alpha \geqq 1, \\ A_2 & \text{if } \alpha < 1. \end{cases} \tag{4.14}$$

Proof. It is well-known that, for an integer $p \geqq 1$

$$\int_0^{\infty} e^{-x^2} x^p dx = \begin{cases} \dfrac{k!}{2} & \text{if } p = 2k+1, \\ \dfrac{(2k-1)!!\sqrt{\pi}}{2^{k+1}} & \text{if } p = 2k. \end{cases}$$

With the help of the inequality

$$\frac{(2k-1)!!}{2^k} \leqq \frac{(2k)!!}{2^k} = k!$$

for each $k \geqq 1$, we can easily see that

$$\int_0^\infty e^{-x^2} x^p dx \leqq \left[\frac{p}{2}\right]! \tag{4.15}$$

for any $p \geqq 1$, where the notation $[x]$ means the largest integer that is less than or equal to $x \in \boldsymbol{R}$. By the fact that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \leqq 1,$$

it turns that (4.15) holds for $p = 0$.

We first consider the case that $\alpha \geqq 1$. Let $m = 2h+1$ for some integer $h \geqq 0$. The left hand side of (4.14) is

$$\sum_{k=0}^\infty \frac{\alpha^{2k}}{(2k)!} \int_0^\infty e^{-x^2} x^{2h+2k+1} dx + \sum_{k=0}^\infty \frac{\alpha^{2k+1}}{(2k+1)!} \int_0^\infty e^{-x^2} x^{2h+2k+2} dx. \tag{4.16}$$

Applying (4.15), the first term of (4.16) is dominated by

$$\sum_{k=0}^\infty \alpha^{2k} \frac{(h+k)!}{(2k)!}. \tag{4.17}$$

Since

$$\frac{(h+k)!}{(2k)!} = \frac{1}{(h+k+1)\cdots(2k)} \leqq \frac{1}{(k-h)!}$$

if $k \geqq h+1$, a bound of (4.17) is

$$\sum_{k=0}^h \alpha^{2k} \frac{(h+k)!}{(2k)!} + \sum_{k=h+1}^\infty \frac{\alpha^{2k}}{(k-h)!} \leqq (h+1)(2h)!\alpha^{2h} + \alpha^{2h} e^{\alpha^2}. \tag{4.18}$$

The second term of (4.16) is dominated by

$$\sum_{k=0}^{\infty} \alpha^{2k+1} \frac{(h+k+1)!}{(2k+1)!}. \tag{4.19}$$

An estimate of (4.19) can be obtained in the same fashion as (4.17). Indeed, with the help of the inequality that

$$\frac{(h+k+1)!}{(2k+1)!} \leqq \frac{1}{(k-h)!}$$

for $k \geqq h+1$, we have that (4.19) is bounded by

$$\sum_{k=0}^{h} \alpha^{2k+1} \frac{(h+k+1)!}{(2k+1)!} + \sum_{n=h+1}^{\infty} \frac{\alpha^{2k+1}}{(k-h)!}, \tag{4.20}$$

which is not larger than

$$(h+1)(2h+1)!\alpha^{2h+1} + \alpha^{2h+1} e^{\alpha^2}.$$

Therefore we obtain (4.14) if m is an odd integer.

If $m = 2h$ for some integer $h \geqq 0$, in virtue of (4.15), the left hand side of (4.14) is dominated by

$$\sum_{k=0}^{\infty} \alpha^{2k} \frac{(h+k)!}{(2k)!} + \sum_{k=0}^{\infty} \alpha^{2k+1} \frac{(h+k)!}{(2k+1)!} \leqq 2\alpha \sum_{k=0}^{\infty} \alpha^{2k} \frac{(h+k)!}{(2k)!},$$

which is not larger than a constant multiple of $\alpha^{2h+1}\exp(\alpha^2)$. This implies (4.14) for $p = 2h$.

In the case that $\alpha < 1$, we may estimate the left hand side of (4.18) if $m = 2h+1$ and (4.20) if $m = 2h$. It is obvious that they are bounded. □

From now on, we will write $C_1, \ldots, C_6$ for suitable constants which are independent of t. We first estimate $\xi_m^{\alpha}(t)$, $\eta_m^{\alpha,\beta}(t)$ and $\zeta_m^{\alpha,\beta}(t)$ in the case that $\alpha > 0$. Since

$$\xi_m^{\alpha}(t) \leqq \sqrt{t}^{m+1} \sum_{n=0}^{\infty} \frac{(2\alpha\sqrt{t})^n}{n!} \int_0^{\infty} e^{-x^2} x^{m+n-1} dx,$$

Lemma 4.4 immediately implies (4.11). Since $(2n-1)!! \leqq (2n+2m-1)!!$, we have that $\eta_m^{\alpha,\beta}(t)$ is dominated by

$$C_1 t^{m+1/2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(2\beta\sqrt{t})^{2n}}{(2n)!} \frac{(2\alpha\sqrt{t})^k}{k!} \int_0^\infty e^{-x^2} x^{2m+2n+k-1} dx. \tag{4.21}$$

By the monotone convergence theorem, the double summation in (4.21) coincides with

$$\int_0^\infty \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(2\beta x\sqrt{t})^{2n}}{(2n)!} \frac{(2\alpha x\sqrt{t})^k}{k!} e^{-x^2} x^{2m-1} dx. \tag{4.22}$$

Note that

$$\sum_{n=1}^{\infty} \frac{(2\beta x\sqrt{t})^{2n}}{(2n)!} \leqq e^{2\beta x\sqrt{t}}.$$

Then (4.22) is not larger than

$$\int_0^\infty e^{-x^2+2(\alpha+\beta)\sqrt{t}x} x^{2m-1} dx = \sum_{n=0}^{\infty} \frac{[2(\alpha+\beta)\sqrt{t}]^n}{n!} \int_0^\infty e^{-x^2} x^{2m+n-1} dx.$$

Therefore we immediately obtain (4.12) by Lemma 4.4. The estimate of $\zeta_m^{\alpha,\beta}(t)$ for $m \geqq 2$ is the same as $\eta_m^{\alpha,\beta}(t)$, and thus the calculation is left to the reader. It remains the estimate of $\zeta_1^{\alpha,\beta}(t)$. We have that

$$\zeta_1^{\alpha,\beta}(t) \leqq C_2 t \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(2\beta\sqrt{t})^{2n}}{(2n)!} \frac{(2\alpha\sqrt{t})^k}{k!} \int_0^\infty e^{-x^2} x^{2n+k} dx.$$

Then the double sum in the right hand side is equal to

$$\int_0^\infty \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(2\beta\sqrt{t})^{2n}}{(2n)!} \frac{(2\alpha\sqrt{t})^k}{k!} e^{-x^2} x^{2n+k} dx \leqq \int_0^\infty e^{-x^2+2(\alpha+\beta)x\sqrt{t}} dx,$$

which coincides with

$$\sum_{n=0}^{\infty} \frac{[2(\alpha+\beta)\sqrt{t}]^n}{n!} \int_0^\infty e^{-x^2} x^n dx.$$

Therefore, by Lemma 4.4, we obtain that (4.21) is not larger than a constant multiple of $t^{3/2}\exp[4(\alpha+\beta)^2 t]$ if $t \geqq 1/4(\alpha+\beta)^2$ and than a constant otherwise.

We next consider the case that $\alpha = 0$. It is easy to estimate $\xi_m^0(t)$. Indeed, we have that

$$\xi_m^0(t) = \frac{\sqrt{t}^{m+1}}{m+1} \int_0^\infty e^{-x^2} x^{m-1} dx.$$

Since $(2n-1)!! \leqq (2n+2m-1)!!$, it follows from Lemma 4.4 that

$$\eta_m^{0,\beta}(t) \leqq C_3 t^{m+1/2} \sum_{n=1}^\infty \frac{(2\beta\sqrt{t})^{2n}}{(2n)!} \int_0^\infty e^{-x^2} x^{2m+2n-1} dx,$$

which is dominated by

$$C_3 t^{m+1/2} \sum_{n=0}^\infty \frac{(2\beta\sqrt{t})^{n}}{n!} \int_0^\infty e^{-x^2} x^{2m+n-1} dx.$$

Therefore, by Lemma 4.4, we conclude that

$$\eta_m^{0,\beta}(t) \leqq C_4 t^{2m+1/2} e^{4\beta^2 t}$$

if $t \geqq 1/4\beta^2$ and that $\eta_m^{0,\beta}(t) \leqq C_5$ if $t < 1/4\beta^2$. This implies (4.12) for $\alpha = 0$. Moreover, we have that

$$\zeta_m^{0,\beta}(t) \leqq C_6 t^m \sum_{n=1}^\infty \frac{(2\beta\sqrt{t})^{2n}}{(2n)!} \int_0^\infty e^{-x^2} x^{2m+2n-2} dx.$$

The summation in the right hand side is dominated by

$$\sum_{n=0}^\infty \frac{(2\beta\sqrt{t})^{n}}{n!} \int_0^\infty e^{-x^2} x^{2m+n-2} dx.$$

Lemma 4.4 yields (4.13) for $\alpha = 0$. We finish proving (4.11), (4.12) and (4.13).

It remains to give the proof of (4.2). Recall that $d = 2\varrho + 3$ and that $h_\varrho(t)$ is represented as a linear combination of $P_N^a(t)$, $Q_N^{a,b}(t)$ and $R_N^{a,b}(t)$. The estimates (4.11), (4.12) and (4.13) yield that there are suitable positive constants κ_1, κ_2, κ_3 and κ_4 such that

$$\sum_{n=1}^\infty \left|\alpha_n^{(d)}\right| t^{n/2} \leqq \kappa_1 t^{\kappa_2} e^{\kappa_3 t}$$

for $t > \kappa_4$. Therefore, we have that $\mathscr{L}[L](\lambda)$ is equal to

$$c_d \sum_{n=1}^{\infty} \alpha_n^{(d)} \int_0^{\infty} e^{-\lambda t} t^{n/2} dt = c_d \sum_{n=1}^{\infty} \frac{\alpha_n^{(d)} \Gamma(n/2+1)}{\lambda^{n/2+1}} \tag{4.23}$$

for $\lambda > \kappa_3$.

Let $\delta = \max\{|\beta_0^{(d)}|, |\beta_1^{(d)}|, \ldots, |\beta_{\varrho+1}^{(d)}|\}$. Then it is obvious that

$$\left|\beta_j^{(d)}\right| \leqq \delta[(2\varrho-1)!!e^2]^j \tag{4.24}$$

for $j = 1, 2, \ldots, \varrho+1$. For simplicity, we write γ for $(2\varrho-1)!!e^2$. Since $\langle n, m\rangle = 0$ if $n < m$, it follows from (4.3) that

$$\sum_{j=2}^{\varrho+2} \frac{\langle \varrho, \varrho+2-j\rangle}{2^{\varrho+2-j}} \beta_j^{(d)} = 0,$$

which yields that

$$\left|\beta_{\varrho+2}^{(d)}\right| \leqq \sum_{j=2}^{\varrho+1} \frac{\langle \varrho, \varrho+2-j\rangle}{2^{\varrho+2-j}} \left|\beta_j^{(d)}\right|.$$

Note that

$$\langle \varrho, \varrho+2-j\rangle = \frac{(\varrho-j+3)\cdots(2\varrho-j+2)}{(j-2)!} \leqq \frac{(\varrho+1)\cdots(2\varrho)}{(j-2)!}$$

for $j = 2, 3, \ldots, \varrho+1$. Then we have that

$$\left|\beta_{\varrho+2}^{(d)}\right| \leqq \sum_{j=2}^{\varrho+1} \frac{(2\varrho)!}{2^{\varrho}\varrho!} \frac{2^{j-2}}{(j-2)!} \cdot \delta\gamma^j \leqq \delta\gamma^{\varrho+1}(2\varrho-1)!! \sum_{j=2}^{\infty} \frac{2^{j-2}}{(j-2)!},$$

which is less than or equal to $\delta\gamma^{\varrho+2}$. We can obtain (4.24) for $j \geqq 1$ inductively. Let $h \geqq 2$ and assume (4.24) for each $j \leqq \varrho+h$. It follows from (4.3) that

$$\sum_{j=h+1}^{\varrho+h+1} \frac{\langle \varrho, \varrho+h+1-j\rangle}{2^{\varrho+h+1-j}} \beta_j^{(d)} = 0.$$

Since

$$\langle \varrho, \varrho + h + 1 - j \rangle \leqq \frac{(\varrho+1)\cdots(2\varrho)}{(j-h-1)!}$$

for $j = h+1, h+2, \ldots, \varrho+h+1$, we have that

$$\left|\beta^{(d)}_{\varrho+h+1}\right| \leqq \sum_{j=h+1}^{\varrho+h} \frac{(2\varrho)!}{2^{\varrho}\varrho!} \frac{2^{j-h-1}}{(j-h-1)!} \cdot \delta\gamma^{j} \leqq \delta\gamma^{\varrho+h+1}.$$

Hence, we can obtain that

$$\sum_{j=0}^{\infty} \beta^{(d)}_{j} x^{j}$$

converges absolutely for $|x| < 1/\gamma$. It follows from (4.3) that

$$\sum_{k=0}^{\infty} \frac{\langle \varrho+1, k\rangle}{2^{k}} x^{k} = \sum_{k=0}^{\infty} \frac{\langle \varrho, k\rangle}{2^{k}} x^{k} \sum_{j=0}^{\infty} \beta^{(d)}_{j} x^{j}.$$

This implies that

$$\frac{K_{\varrho+3/2}(r\sqrt{2\lambda})}{K_{\varrho+1/2}(r\sqrt{2\lambda})} = \sum_{j=0}^{\infty} \frac{\beta^{(d)}_{j}}{(r\sqrt{2\lambda})^{j}}$$

for $\lambda > 2r^2/\gamma^2$. Hence, by (4.6), we obtain that $\mathscr{L}[L](\lambda)$ is equal to

$$c_d \sum_{j=1}^{\infty} \frac{\beta^{(d)}_{j-1}}{(\sqrt{2}r)^{j-1}} \frac{1}{\lambda^{j/2+1}} \tag{4.25}$$

for $\lambda > 2r^2/\gamma^2$. Comparing coefficients of (4.25) and the right hand side of (4.23), we can conclude (4.2). This completes the proof of Theorem 4.1.

5. Proof of Lemma 4.3.

In this section, we will give a proof of Lemma 4.3 provided in the previous section. We first give explicit forms of $q^{a,b}_{N}$ and $r^{a,b}_{N}$. It follows from the general formulae of inverse Laplace transforms

$$\mathscr{L}^{-1}[q_N^{a,b}](t) = \frac{\sqrt{\pi}e^{at}}{b^{N-1/2}\Gamma(N)}\left(\frac{1}{2}t\right)^{N-1/2} J_{N-1/2}(bt),$$

$$\mathscr{L}^{-1}[r_N^{a,b}](t) = \frac{\sqrt{\pi}e^{at}}{b^{N-3/2}\Gamma(N)}\left(\frac{1}{2}t\right)^{N-1/2} J_{N-3/2}(bt),$$

where J_μ is the Bessel function of order μ. The Poisson expression of Bessel functions,

$$J_\mu(x) = \frac{2}{\sqrt{\pi}\Gamma(\mu+1/2)}\left(\frac{1}{2}x\right)^{\mu}\int_0^1 (1-v^2)^{\mu-1/2}\cos(xv)dv \tag{5.1}$$

for $\mu \in \boldsymbol{C}$ satisfying with $\operatorname{Re}\mu > -1/2$, is quite useful for calculating $\mathscr{L}^{-1}[q_N^{a,b}]$ and $\mathscr{L}^{-1}[r_N^{a,b}]$. Applying (5.1) for $\mu = N - 1/2$, we have that, for $N \geqq 1$

$$\mathscr{L}^{-1}[q_N^{a,b}](t) = \frac{e^{at}t^{2N-1}}{2^{2N-2}\Gamma(N)^2}\int_0^1 (1-v^2)^{N-1}\cos(btv)dv,$$

which coincides with

$$\frac{e^{at}t^{2N-1}}{2^{2N-2}\Gamma(N)^2}\int_0^1 (1-v^2)^{N-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(btv)^{2n}dv. \tag{5.2}$$

Since

$$\sum_{n=0}^{\infty}\int_0^1 (1-v^2)^{N-1}\frac{1}{(2n)!}(btv)^{2n}dv \leqq \sum_{n=0}^{\infty}\frac{(bt)^{2n}}{(2n)!} \leqq e^{bt},$$

we can change the order of integral and summation in (5.2). Then $\mathscr{L}^{-1}[q_N^{a,b}](t)$ is equal to

$$\frac{e^{at}t^{2N-1}}{2^{2N-2}\Gamma(N)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(bt)^{2n}}{(2n)!}\int_0^1 (1-v^2)^{N-1}v^{2n}dv.$$

The formula

$$\int_0^1 (1-v^2)^n v^m dv = \begin{cases} \dfrac{2^n n!(m-1)!!}{(m+2n+1)!!} & \text{if } m \geqq 1, \\ \dfrac{2^n n!}{(2n+1)!!} & \text{if } m = 1 \end{cases}$$

for an integer $n \geqq 1$ immediately implies that

$$\mathscr{L}^{-1}[q_N^{a,b}](t) = \frac{e^{at}t^{2N-1}}{2^{N-1}\Gamma(N)}\left[\sum_{n=1}^{\infty}\frac{(-1)^n(bt)^{2n}}{(2n)!}\frac{(2n-1)!!}{(2n+2N-1)!!}+\frac{1}{(2N-1)!!}\right]. \tag{5.3}$$

By the same calculation, we have that, for $N \geqq 2$

$$\mathscr{L}^{-1}[r_N^{a,b}](t) = \frac{e^{at}t^{2N-2}}{2^{N-1}\Gamma(N)}\left[\sum_{n=1}^{\infty}\frac{(-1)^n(bt)^{2n}}{(2n)!}\frac{(2n-1)!!}{(2n+2N-3)!!}+\frac{1}{(2N-3)!!}\right]. \tag{5.4}$$

The proof is left to the reader. Since $J_{-1/2}$ does not have the Poisson expression like (5.1), the same computation can not be applied to $r_1(t)$ unfortunately. However, we can calculate it directly, and then have that

$$\mathscr{L}^{-1}[r_1^{a,b}](t) = e^{at}\cos(bt) = e^{at}\sum_{n=0}^{\infty}\frac{(-1)^n(bt)^{2n}}{(2n)!}. \tag{5.5}$$

We are now ready to show (4.7) and (4.8). For $N \geqq 1$ it follows from (5.3) that $Q_N^{a,b}(t)$ is equal to

$$\int_0^t \frac{1}{\sqrt{w}}\left[\int_0^{\infty} e^{-u^2/4w}\frac{e^{au}u^{2N-1}}{2^{N-1}\Gamma(N)} \times \sum_{n=1}^{\infty}\frac{(-1)^n(bu)^{2n}}{(2n)!}\frac{(2n-1)!!}{(2n+2N-1)!!}du\right]dw \tag{5.6}$$

$$+\int_0^t \frac{1}{\sqrt{w}}\left[\int_0^{\infty} e^{-u^2/4w}\frac{e^{au}u^{2N-1}}{2^{N-1}\Gamma(N)}\frac{1}{(2N-1)!!}du\right]dw. \tag{5.7}$$

In order to change the order of the double integral and the summation in (5.6), it is necessary to see the convergence of

$$\int_0^t \frac{1}{\sqrt{w}}\left[\int_0^{\infty} e^{-u^2/4w+au}u^{2N-1}\sum_{n=1}^{\infty}\frac{(bu)^{2n}}{(2n)!}\frac{(2n-1)!!}{(2n+2N-1)!!}du\right]dw. \tag{5.8}$$

Since $(2n-1)!! \leqq (2n+2N-1)!!$ for $n \geqq 1$, it is easy to obtain that (5.8) is bounded by $\gamma_{2N-1}^{a+b}(t)$, which is equal to

$$2^{N+1}\int_0^{\sqrt{t}}\left[\int_0^\infty e^{-x^2+2(a+b)xy}x^{2N-1}y^{2N}dx\right]dy<\infty.$$

Therefore, (5.6) is equal to

$$\frac{1}{2^{N-1}\Gamma(N)}\sum_{n=1}^\infty\frac{(-1)^n b^{2n}}{(2n)!}\frac{(2n-1)!!}{(2n+2N-1)!!}$$
$$\times\int_0^t\frac{1}{\sqrt{w}}\left[\int_0^\infty e^{-u^2/4w+au}u^{2N+2n-1}du\right]dw.$$

This coincides with the second term of the right hand side of (4.7). It follows that (5.7) is equal to

$$\frac{\gamma_{2N-1}^a(t)}{2^{N-1}(N-1)!(2N-1)!!}=\frac{\gamma_{2N-1}^a(t)}{(2N-1)!}=P_{2N-1}^a(t),$$

which is the first term of the right hand side of (4.7).

The computations of (5.4) and (5.5) are the same, and thus details of the proof of (4.8) are left to the reader.

6. The five and seven dimensional cases.

This section is devoted to giving the formula for $L(t)$ when d is five or seven. For convenience, let $\beta=1/\sqrt{2r^2}$. If $d=5$, it follows from (4.5) that

$$\mathscr{L}[L](\lambda)=\frac{c_5}{\sqrt{\lambda}^3}\frac{K_{5/2}(\sqrt{\lambda}/\beta)}{K_{3/2}(\sqrt{\lambda}/\beta)}.$$

With the help of (2.7), we have that

$$\frac{K_{5/2}(x)}{K_{3/2}(x)}=1+\frac{3}{x}-\frac{1}{x+1}.$$

This implies that $\mathscr{L}[L](\lambda)$ is

$$c_5\left[\frac{1}{\sqrt{\lambda}^3}+\frac{3\beta}{\lambda^2}-\frac{\beta}{\sqrt{\lambda}^3(\sqrt{\lambda}+\beta)}\right]=\frac{S_4r^4}{\sqrt{2}}\left[\frac{3\beta}{\lambda^2}+\frac{1}{\beta\lambda}-\frac{1}{\beta}\frac{1}{\sqrt{\lambda}(\sqrt{\lambda}+\beta)}\right].$$

The inverse Laplace transform of $1/\sqrt{\lambda}(\sqrt{\lambda}+\beta)$ is $\exp(\beta^2 t)\,\mathrm{erfc}(\beta\sqrt{t})$, where

$$\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy$$

for $z \in \boldsymbol{C}$. Therefore we obtain the following theorem.

THEOREM 6.1. *If $d = 5$, we have that*

$$L(t) = S_4 r^4 \left[\frac{3t}{2r} + r - r \exp\left(\frac{t}{2r^2}\right) \mathrm{erfc}\left(\sqrt{\frac{t}{2r^2}}\right) \right].$$

The computation in the seven dimensional case is similar but more complicated. If $d = 7$, it follows from (2.7) and (4.5) that $\mathscr{L}[L](\lambda)$ is equal to

$$\frac{c_7}{\sqrt{\lambda}^3} \frac{K_{7/2}(\sqrt{\lambda}/\beta)}{K_{5/2}(\sqrt{\lambda}/\beta)} = c_7 \left(\frac{5\beta}{\lambda^2} + \frac{1}{3\beta\lambda} - \frac{1}{3\beta} \frac{1}{\lambda + 3\beta\sqrt{\lambda} + 3\beta^2} \right).$$

Therefore we need to give the inverse Laplace transform of

$$\frac{1}{\lambda + 3\beta\sqrt{\lambda} + 3\beta^2}.$$

We write $f(\lambda)$ for this expression. Then we have that

$$f(\lambda) = \frac{1}{(\sqrt{\lambda} + a)^2 + b^2},$$

where $a = 3\beta/2$ and $b = \sqrt{3}\beta/2$. Let $g(\lambda) = f(\lambda^2)$. Since

$$\mathscr{L}^{-1}[f](t) = \frac{1}{\sqrt{2\pi t^3}} \int_0^\infty e^{-w^2/4t} w \mathscr{L}^{-1}[g](w) dw,$$
$$\mathscr{L}^{-1}[g](t) = \frac{1}{b} e^{-at} \sin(bt),$$

then we have that

$$\mathscr{L}^{-1}[f](t) = \frac{1}{b\sqrt{2\pi t^3}} \int_0^\infty e^{-w^2/4t} w e^{-aw} \sin(bw) dw. \tag{6.1}$$

We can find in Prudnikov-Brychkov-Marichev [**12**, p. 90] that the integral in the right hand side of (6.1) is

$$-i\sqrt{\pi t^3}\left[(a+bi)e^{(a+bi)^2 t}\operatorname{erfc}((a+bi)\sqrt{t})-(a-bi)e^{(a-bi)^2 t}\operatorname{erfc}((a-bi)\sqrt{t})\right].$$

Then the simple calculation shows the following theorem.

THEOREM 6.2. *If $d=7$, we have that*

$$\begin{aligned}L(t) &= S_6 r^6\left[\frac{5t}{2r}+\frac{r}{3}-\frac{2r^2}{3\sqrt{3\pi t^3}}\int_0^\infty \exp\left(-\frac{x^2}{4t}-ax\right)x\sin(bx)dx\right]\\ &= S_6 r^6\left[\frac{5t}{2r}+\frac{r}{3}-\left(\frac{1}{\sqrt{6}}-i\right)\exp\left(\frac{8+3\sqrt{6}\,i}{8r^2}t\right)\operatorname{erfc}\left(\frac{3\sqrt{2}+\sqrt{3}\,i}{4r}\sqrt{t}\right)\right.\\ &\qquad\left.-\left(\frac{1}{\sqrt{6}}+i\right)\exp\left(\frac{8-3\sqrt{6}\,i}{8r^2}t\right)\operatorname{erfc}\left(\frac{3\sqrt{2}-\sqrt{3}\,i}{4r}\sqrt{t}\right)\right].\end{aligned}$$

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