Modified generalized Hadamard matrices and constructions for transversal designs

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Abstract. It is well known that there exists a transversal design $\mathrm{TD}_\lambda[k;u]$ admitting a class regular automorphism group U if and only if there exists a generalized Hadamard matrix $\mathrm{GH}(u,\lambda)$ over U. Note that in this case the resulting transversal design is symmetric by Jungnickel's result.

In this article we define a *modified generalized Hadamard matrix* and show that transversal designs which are not necessarily symmetric can be constructed from these under a modified condition similar to class regularity even if it admits no class regular automorphism group.

Keywords: transversal design, generalized Hadamard matrix, group, spread

1 Introduction

A *transversal design* $\mathrm{TD}_\lambda[k;u]$ $(u>1, k=u\lambda)$ is an incidence structure $(\mathbb{P},\mathbb{B})$, where

(i) $\mathbb{P}$ is a set of uk points partitioned into k classes (called *point classes*), each of size u,

(ii) $\mathbb{B}$ is a collection of k-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no block and any two points in distinct point classes are incident with exactly λ blocks.

A transversal design $\mathcal{D}=(\mathbb{P},\mathbb{B})$ is called *symmetric* if the dual structure $\mathcal{D}^*$ of $\mathcal{D}$ is also a transversal design with the same parameters as $\mathcal{D}$. If $\mathcal{D}$ is symmetric, the point classes of $\mathcal{D}^*$ are said to be the *block classes* of $\mathcal{D}$.

A transversal design $\mathcal{D}$ is called *class regular* with respect to U if U is an automorphism group of $\mathcal{D}$ acting regularly on each point class. If $\mathcal{D}$ is class regular with respect to U, then there exists a *generalized Hadamard matrix* $[d_{i,j}]$ of order k with entries from U (for short $\mathrm{GH}(u,\lambda)$) such that whenever $i\neq\ell$ the set of differences $\{d_{ij}d_{\ell j}^{-1} \mid 1\leq j\leq k\}$ contains each element of U exactly λ times. Conversely, from a generalized Hadamard matrix $\mathrm{GH}(u,\lambda)$ over a group U of order u, one can construct a transversal design $\mathrm{TD}_\lambda[k;u]$ which admits U as a class regular automorphism group (Theorem 3.6 of [2]).

In this case $(\mathbb{P}, \mathbb{B})$ is necessarily a symmetric transversal design by a result of Jungnickel (Corollary 6.9 of [7]). Hence, if a transversal design $\mathrm{TD}_\lambda[k; u]$ is not symmetric, it admits no class regular automorphism group and so can not be obtained from any generalized Hadamard matrix.

In this article we show that if $(\mathbb{P}, \mathbb{B})$ admits an automorphism group G acting semiregularly on $\mathbb{P} \cup \mathbb{B}$ and if for each point class $\mathcal{C}$ there exists a subgroup $U_\mathcal{C} (\leq G)$ acting regularly on $\mathcal{C}$, then we can obtain a *modified generalized Hadamard matrix* (Theorem 3.2, Definition 3.3). $U_\mathcal{C}$ is, so to speak, an individual class regular subgroup depending on $\mathcal{C}$. We note that $U_\mathcal{C}$ is not always a normal subgroup of G. Conversely, we show that from a modified generalized Hadamard matrix we can construct a transversal design which does not always admit a class regular automorphism group (Theorem 3.5). As an example we construct transversal designs from any translation plane using this method (Theorem 4.1). These are not always admit class regular automorphism groups (see Remark 4.2). Furthermore, by a *modified Kronecker product* we construct many transversal designs (Theorem 5.2).

2 Preliminaries

Let H be a group of order $q^2 (> 1)$. A set of $q + 1$ subgroups $\{H_1, \cdots, H_{q+1}\}$ of H is called a *spread* of H if $|H_1| = \cdots = |H_{q+1}| = q$ and $H_i \cap H_j = 1$ for all distinct i and j with $1 \leq i, j \leq q + 1$. Then the following holds.

Lemma 2.1. ([6],[8]) Let H be a group of order q^2 and $\{H_1, \cdots, H_{q+1}\}$ a spread of H. Then

(i) $H = H_i H_j$ for each i and j, $1 \leq i \neq j \leq q + 1$.

(ii) $H^* = H_1^* \cup \cdots \cup H_{q+1}^*$ is a disjoint union, where $X^* = X \setminus \{1\}$ for a subgroup X of H.

(iii) H is an elementary abelian p-group for a prime p.

The following fact is well known.

Lemma 2.2. Let $\mathcal{D} = (\mathbb{P}, \mathbb{B})$ be a transversal design $\mathrm{TD}_\lambda[k; u]$ and let U be a class regular automorphism group of $\mathcal{D}$. Then $\mathcal{D}$ is symmetric and U is also a class regular automorphism group of the dual $\mathcal{D}^*$ of $\mathcal{D}$.

We note the following simple fact without proof, which we often use for the rest of the paper .

Lemma 2.3. Let H and N be subgroups of a group G. If $G = HN$ (as the product operation on G), then $\sum_{x \in H} \sum_{x \in N} = |H \cap N| \sum_{x \in G}$ as multiplication in the group ring $\mathbb{Z}[G]$.

Let $(\mathbb{P}, \mathbb{B})$ be a transversal design $\mathrm{TD}_\lambda[k; u]$ with $k = u\lambda$ and $u > 1$. For points $P, Q \in \mathbb{P}$, $P \sim Q$ indicates that P and Q are in the same point class of $(\mathbb{P}, \mathbb{B})$.

Let G be a group. For a subset S of G, $S^{(-1)} = \{x^{-1} : x \in S\}$. Similarly, for a group ring element $z = \sum_{x\in G} a_x x \in \mathbb{Z}[G]$, $z^{(-1)} = \sum_{x\in G} a_x x^{-1}$ $(\in \mathbb{Z}[G])$. Throughout the rest of this paper, all sets and groups are assumed to be finite and a subset S of a group G is identified with a group ring element $\sum_{x\in S} x$ $(\in \mathbb{Z}[G])$ unless specifically stated.

3 Modified generalized Hadamard matrices

In this section we consider a transversal design which admits an automorphism group satisfying the following.

Hypothesis 3.1. Let $(\mathbb{P}, \mathbb{B})$ be a transversal design $\mathrm{TD}_\lambda[k; u]$ and let H be an automorphism group of $(\mathbb{P}, \mathbb{B})$ such that

(i) H acts semiregularly both on $\mathbb{P}$ and on $\mathbb{B}$,

(ii) Each H-orbit on $\mathbb{P}$ is a union of some point classes.

From Hypothesis 3.1, $|H| = us$ for some integer $s \mid u\lambda$ and each H-orbit on $\mathbb{P}$ contains exactly s point classes. Moreover, setting $t = \frac{k}{s}$, H has exactly t orbits both on $\mathbb{P}$ and on $\mathbb{B}$.

Theorem 3.2. *Assume Hypothesis 3.1. Let $\{\mathbb{Q}_1, \cdots, \mathbb{Q}_t\}$ be the set of H-orbits on $\mathbb{P}$ and $\{\mathbb{B}_1, \cdots, \mathbb{B}_t\}$ the set of H-orbits on $\mathbb{B}$. If we choose $Q_i \in \mathbb{Q}_i$ and $B_i \in \mathbb{B}_i$ for each i, $1 \le i \le t$, then the following holds.*

(i) *Set $U_i = \{x \in H : Q_i x \sim Q_i\}$, $1 \le i \le t$. Then U_i is a subgroup of H of order u.*

(ii) *Set $D_{ij} = \{x \in H : Q_i x \in B_j\}$, $1 \le i, j \le t$. Then,*

  (a) $|D_{ij}| = s \quad (1 \le i, j \le t)$.

  (b) $\sum_{j=1}^{t} D_{ij} D_{\ell j}^{(-1)} = \lambda H \quad (1 \le i \ne \ell \le t)$.

  (c) $\sum_{j=1}^{t} D_{ij} D_{ij}^{(-1)} = k + \lambda(H - U_i) \quad (1 \le i \le t)$.

Proof. Let $\mathbb{P}_r (\subset \mathbb{Q}_i)$ be the point class containing the point Q_i. As H acts on $\mathbb{Q}_i$ regularly, $|U_i| = |\mathbb{P}_r| = u$. Let $x, y \in U_i$. Then $Q_i\, x \sim Q_i$ and $Q_i\, y \sim Q_i$. From this $Q_i\, x \sim Q_i\, y$. Hence $Q_i\, xy^{-1} \sim Q_i$ and so $xy^{-1} \in U_i$. Thus (i) holds.

We fix i and ℓ $(1 \le i, \ell \le t)$ and assume first that $i \ne \ell$. For each $c \in H$ we set $\Gamma_{i,\ell,c} = \{(a, b, j) : a \in D_{ij}, b \in D_{\ell j},\ 1 \le j \le t,\ c = ab^{-1}\}$. It suffices to prove $|\Gamma_{i,\ell,c}| = \lambda$ in order to check (b). The condition on $\Gamma_{i,\ell,c}$ is equivalent to $c = ab^{-1}$ and $Q_i a, Q_\ell b \in B_j$. Hence $Q_i c, Q_\ell \in B_j b^{-1}$. As $Q_i c \in \mathbb{Q}_i$, $Q_\ell \in \mathbb{Q}_\ell$ and $i \ne \ell$, it follows that $Q_i c \not\sim Q_\ell$ and so there exist exactly λ blocks containing both $Q_i c$ and Q_ℓ, say $B'_1, \cdots, B'_\lambda$. For each m $(1 \le m \le \lambda)$, there exists a unique pair (j, b) such that $B'_m = B_j b^{-1}, 1 \le j \le t,\ b \in H$. Since $Q_\ell \in B'_m = B_j b^{-1}$, we have $Q_\ell b \in B_j$. From this, $b \in D_{\ell j}$. Moreover, $Q_i c \in B'_m = B_j b^{-1}$. Hence $Q_i a \in B_j$ and so $a \in D_{ij}$. Therefore $|\Gamma_{i,\ell,c}| = \lambda$ and (b) holds.

We now consider $\Gamma_{i,i,c} = \{(a,b,j) : a \in D_{ij}, b \in D_{ij},\ 1 \le j \le t,\ c = ab^{-1}\}$. Similarly as in the proof of (b), the condition is equivalent to $Q_i c, Q_i \in B_j b^{-1}$. If $c = 1$, the number of blocks containing Q_i is exactly k. Let $B'_1, \cdots, B'_k$ be such blocks. Then, for each m, $1 \le m \le k$, there exists a unique (j,b) such that $B'_m = B_j b^{-1}, 1 \le j \le t,\ b \in H$. As $Q_i \in B_j b^{-1}$, we have $b \in D_{ij}$. Hence $|\Gamma_{i,i,1}| = k$. Assume $c \in U_i \setminus \{1\}$. Then, as $Q_i \sim Q_i c$ and $Q_i \neq Q_i c$, $|\Gamma_{i,i,c}| = 0$. Assume $c \in H \setminus U_i$. Then, as $Q_i \not\sim Q_i c$, the number of blocks containing Q_i and $Q_i c$ is λ. Let $B'_1, \cdots, B'_\lambda$ be such blocks. For each m $(1 \le m \le \lambda)$ there exists a unique (j,b) such that $B'_m = B_j b^{-1}, 1 \le j \le t,\ b \in H$. Since $Q_i, Q_i c \in B'_m = B_j b^{-1}$, $b \in D_{ij}$ and $a \in D_{ij}$. Hence $|\Gamma_{i,i,c}| = \lambda$. Thus (c) holds.

By (c), $D_{ij} D_{ij}^{(-1)} \cap U_i \setminus \{1\} = \emptyset$. Hence, as $|H| = su$ and $|U_i| = u$, we have $|D_{ij}| \le s$ counting cosets of U_i. Applying the trivial character χ_0 of H to (c), we have $s^2 t \ge \sum_{1 \le j \le t} |D_{ij}|^2 = \chi_0(k + \lambda(H - U_i)) = s^2 t$. This forces $|D_{ij}| = s$ for each $i, j,\ 1 \le i, j \le t$. Thus (a) holds. □

We show that the converse of Theorem 3.2 is also true. We first generalize the notion of a generalized Hadamard matrix. The set of n by n matrices with coefficients in a ring R is denoted by $\mathrm{M}_n(R)$.

Definition 3.3. Let H be a group of order su. For subsets D_{ij} $(1 \le i, j \le t, st = u\lambda)$ of H, we call a matrix $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[H])$ *a modified generalized Hadamard matrix* with respect to subgroups U_i $(1 \le i \le t)$ of H of order u if the following conditions are satisfied :

$|D_{ij}| = s$ for all $i, j,\ 1 \le i, j \le t$, and

$$\sum_{1 \le j \le t} D_{ij} D_{\ell j}^{(-1)} = \begin{cases} k + \lambda(H - U_i) & \text{if } i = \ell, \\ \lambda H & \text{otherwise.} \end{cases} \tag{1}$$

For short, we say $[D_{ij}]$ is a *GH*(s,u,λ) *matrix with respect to* U_i, $1 \le i \le t$. If $U_1 = \cdots = U_t = U$ for a subgroup U of H, we simply say that $[D_{ij}]$ is a $\mathrm{GH}(s,u,\lambda)$ matrix with respect to U.

Remark 3.4. If $[D_{ij}]$ is a $\mathrm{GH}(s,u,\lambda)$ matrix with respect to a normal subgroup U of H, the notion in Definition 3.3 is the same as that of [1]. Clearly any $\mathrm{GH}(1,u,\lambda)$ matrix is an ordinary generalized Hadamard matrix $\mathrm{GH}(u,\lambda)$ (see [2]). Moreover, if D is a $(u\lambda, u, u\lambda, \lambda)$-difference set (see [9]) in a group G relative to U, then the 1 by 1 matrix $[D]$ is a $\mathrm{GH}(u\lambda, u, \lambda)$ matrix with respect to U.

We now show that a transversal design $\mathrm{TD}_\lambda[k;u]$ is obtained from a $\mathrm{GH}(s,u,\lambda)$ matrix. For a $\mathrm{GH}(s,u,\lambda)$ matrix $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[H])$, we define a set of points $\mathbb{P}$ and a set of blocks $\mathbb{B}$ in the following way.

$$\mathbb{P} = \{1, 2, \cdots, t\} \times H, \quad \mathbb{B} = \{B_{jh} : 1 \le j \le t,\ h \in H\}, \tag{2}$$
$$\text{where } B_{jh} = \bigcup_{1 \le i \le t} (i, D_{ij} h) \ (= \bigcup_{1 \le i \le t} \{(i, dh) : 1 \le i \le t,\ d \in D_{ij}\}).$$

Then we have

Theorem 3.5. *Let $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[H])$ be a GH(s,u,λ) matrix over a group H of order su with respect to subgroups U_i $(1 \le i \le t)$, where $t = u\lambda/s$. If we define $\mathbb{P}$ and $\mathbb{B}$ by (2), then the following holds.*

(i) *$(\mathbb{P},\mathbb{B})$ is a transversal design $\mathrm{TD}_\lambda[k;u]$ $(k = u\lambda)$.*

(ii) *For each i $(1 \le i \le t)$ and $x \in H$, set $\mathbb{P}_{i,U_ix} = \{(i,wx) : w \in U_i\}$ $(1 \le i \le t, x \in H)$. Then $\mathbb{P}_{i,U_ix}$ is a point class of $(\mathbb{P},\mathbb{B})$.*

(iii) *If we define the action of H on $(\mathbb{P},\mathbb{B})$ by $(i,c)^x = (i,cx), (B_{j,d})^x = B_{j,dx}$, then H is an automorphism group of $(\mathbb{P},\mathbb{B})$ acting semiregularly both on $\mathbb{P}$ and on $\mathbb{B}$.*

(iv) *For every $x \in H$, $x^{-1}U_ix$ acts regularly on a point class $\mathbb{P}_{i,U_ix}$ $(1 \le i \le t)$.*

Proof. Clearly each block contains exactly k points. We choose two distinct points $(i,a),(\ell,b) \in \mathbb{P}$ and count the number of blocks $N_{(i,a),(\ell,b)}$ containing both (i,a) and (ℓ,b) :

$$N_{(i,a),(\ell,b)} = |\{B_{jh} : ah^{-1} \in D_{ij},\ bh^{-1} \in D_{\ell j}\ 1 \le j \le t,\ h \in H\}|.$$

Put $N = N_{(i,a),(\ell,b)}$ and $d_1 = ah^{-1}$, $d_2 = bh^{-1}$. Then we have

$$N = |\{(j,d_1,d_2) : 1 \le j \le t,\ d_1 \in D_{ij},\ d_2 \in D_{\ell j},\ d_1d_2^{-1} = ab^{-1}\}|.$$

Assume $i = \ell$. Then $a \ne b$. By (1), $N = 0$ if $ab^{-1} \in U_i \setminus \{1\}$ and $N = \lambda$ if $ab^{-1} \in H \setminus U_i$. On the other hand, assume $i \ne \ell$. Then, as $d_1d_2^{-1} = ab^{-1} \in H$, we have $N = \lambda$ by (1). Therefore (i) and (ii) hold.

By the definition of the action of H on $(\mathbb{P},\mathbb{B})$,

$$\begin{aligned} &(i,c) \in B_{j,d} \iff (i,c) \in (i, D_{ij}d) \iff c \in D_{ij}d \\ &\iff cx \in D_{ij}dx \iff (i,cx) \in B_{j,dx} \iff (i,c)^x \in B^x_{jd}. \end{aligned}$$

Thus (iii) holds. Moreover, as $(\mathbb{P}_{i,U_ix})x^{-1}U_ix = \mathbb{P}_{i,U_ix(x^{-1}U_ix)} = \mathbb{P}_{i,U_ix}$, (iv) also holds. $\square$

Example 3.6. Let $u = 3$ and $\lambda = 2$. Let $H = \langle a,b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ be an abelian group of order 9 generated by a and b and set $s = 3$ and $t = 2$. We put $D_{ij} \in \mathbb{Z}[H],\ 1 \le i,j \le 2$, and $U_i \simeq \mathbb{Z}_3,\ 1 \le i \le 2$, as follows :

$$U_1 = \langle b\rangle, \quad U_2 = \langle a\rangle$$

$$[D_{ij}] = \begin{bmatrix} 1+ab+a^2b & a^2+b+ab \\ 1+ab+ab^2 & 1+a^2b^2+a^2b \end{bmatrix}$$

Then we can verify that D_{ij} and U_j satisfy the following.

$$
\begin{aligned}
D_{11}D_{11}^{(-1)} + D_{12}D_{12}^{(-1)} &= 6 + 2(H - U_1) \\
D_{21}D_{21}^{(-1)} + D_{22}D_{22}^{(-1)} &= 6 + 2(H - U_2) \\
D_{11}D_{21}^{(-1)} + D_{12}D_{22}^{(-1)} &= 2H
\end{aligned}
$$

Therefore $[D_{ij}]$ is a $\mathrm{GH}(3,3,2)$ matrix with respect to U_1, U_2 and applying Theorem 3.5 we obtain a transversal design $\mathrm{TD}_2[6;3]$.

Example 3.7. Set $u = 3$, $\lambda = 4$ and $H = \langle a, b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_6$. Set $t = 2$ and $s = 6$. We put D_{ij} $(1 \le i, j \le 2)$ and $U_i \simeq \mathbb{Z}_3$ $(1 \le i \le 2)$ as follows.

$$U_1 = \langle ab^2\rangle, \quad U_2 = \langle ab^4\rangle$$

$$[D_{ij}] = \begin{bmatrix} 1 + b + b^2 + b^3 + a + ab & 1 + a^2b^5 + ab^4 + a^2b + b^4 + ab \\ 1 + ab^5 + a^2b^4 + ab + b^4 + a^2b & 1 + b + b^2 + b^3 + a^2 + a^2b \end{bmatrix}$$

Then we can check that D_{ij} and U_j satisfy (1). Hence $[D_{ij}]$ is a $\mathrm{GH}(6,3,4)$ matrix with respect to U_1 and U_2 and by Theorem 3.5 we obtain a $\mathrm{TD}_4[12;3]$. Actually, $[D_{ij}]$ is constructed by using a non-normal $(12,3,12,4)$-difference set in [5]. We mention the method of construction of $\mathrm{GH}(s,u,\lambda)$ matrices from $(u\lambda, u, u\lambda, \lambda)$-difference sets in Section 6.

Remark 3.8. It is possible to have $H \rhd U_i$ for every i, $1 \le i \le t$. But, it is not always true that $U_1 = \cdots = U_t$. The group $\langle U_1, \cdots, U_t\rangle$ generated by U_i's is kind of like a minimal group in order to determine the transversal design even if there is no class regular automorphism group.

Lemma 3.9. Let $[D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[H])$ be a $\mathrm{GH}(s,u,\lambda)$ matrix over a group H, $t = u\lambda/s$. Let $j, \ell \in \{1, 2, \cdots, t\}$ and let $a, b \in H$. If $D_{ij}a = D_{i\ell}b$ for every i, then $(j,a) = (\ell, b)$. In particular, no two columns of a $\mathrm{GH}(s,u,\lambda)$ matrix are identical.

Proof. Assume $D_{ij}a = D_{i\ell}b$ for every i, $1 \le i \le t$ and let $(\mathbb{P}, \mathbb{B})$ be the transversal design $\mathrm{TD}_\lambda[k;u]$ constructed in Theorem 3.5. Then $B_{ja} = B_{\ell b}$ as subsets of $\mathbb{P}$. On the other hand, by Lemma 1.10 of [7], any transversal design $\mathrm{TD}_\lambda[k;u]$ with $u > 1$ has no repeated blocks. Hence $B_{ja} = B_{\ell b}$ implies $(j,a) = (\ell, b)$. Thus the lemma holds. □

Theorem 3.10. *Let $[D_{ij}]$ be a GH(s,u,λ) matrix over a group H with respect to subgroups U_i of H, $1 \le i \le t = u\lambda/s$. Then the transversal design TD$_\lambda[k;u], k = u\lambda$, corresponding to $[D_{ij}]$ is symmetric if and only if $[D_{ij}^{(-1)}]^T$ is a GH(s,u,λ) matrix over H with respect to suitable subgroups V_i of H, $1 \le i \le t$. If this condition is satisfied, then B_{j,V_jx} is a block class for all j and x, $1 \le j \le t$, $x \in H$.*

Proof. Assume the transversal design $(\mathbb{P}, \mathbb{B})$ corresponding to $[D_{ij}]$ is symmetric. By Lemma 3.9, $B_{jg} \ne B_{\ell h}$ if $(j,g) \ne (\ell, h)$. We note that

(a) If $(j,g)\neq(\ell,h)$ and $B_{jg}\sim B_{\ell h}$, then $\bigcup_{1\le i\le t}(i,D_{ij}g)\cap\bigcup_{1\le i\le t}(i,D_{i\ell}h)=\emptyset$.
(b) If $B_{jg}\not\sim B_{\ell h}$, then $|\bigcup_{1\le i\le t}(i,D_{ij}g)\ \cap\ \bigcup_{1\le i\le t}(i,D_{i\ell}h)|=\lambda$.

The facts (a) and (b) imply that each coefficient of $\sum_{1\le i\le t}D_{ij}^{(-1)}D_{i\ell}$ is either 0 or λ. As $(j,g)\neq(\ell,h)$ iff either $j\neq\ell$ or $j=\ell$ and $gh^{-1}\neq 1$, there are subsets S of H and T of $H\setminus\{1\}$ such that

$$\sum_{1\le i\le t}D_{ij}^{(-1)}D_{i\ell} = \lambda S\ \ (j\neq\ell) \tag{3}$$

$$\sum_{1\le i\le t}D_{ij}^{(-1)}D_{ij} = k+\lambda T \tag{4}$$

We will show that (i) $S=H$ and (ii) $T=H-V_j$ for a subgroup of H of order u.

First we show (i). Applying the trivial character of H to (3) $ts^2=\lambda|S|$. Hence $|S|=sk/\lambda=us=|H|$, which implies $S=H$.

We now show (ii). Set $V_j=H\setminus T$. Then, applying the trivial character of H to (4) we have $k+\lambda|T|=ts^2$. Hence $|T|=|H|-u$ and so $|V_j|=u$. Moreover, we show that V_j is a subgroup of H of order u. We first note that $1\in V_j$. Clearly $V_j\setminus\{1\}=\{x\in H: B_{j,x}\cap B_{j,1}=\emptyset\}$. Let $x,y\in V_j\setminus\{1\}, x\neq y$. Then, $B_{j,1}\cap B_{j,x}=\emptyset$ and $B_{j,1}\cap B_{j,y}=\emptyset$. Hence $B_{j,x}\sim B_{j,1}\sim B_{j,y}$ and so $B_{j,x}\sim B_{j,y}$. It follows that $B_{j,xy^{-1}}\cap B_{j,1}=\emptyset$. From this, $xy^{-1}\in V_j\setminus\{1\}$. Hence V_j is a subgroup of H of order u and $T=H-V_j$. Consequently,

$$\sum_{i=1}^{t}D_{ij}^{(-1)}D_{i\ell}=\begin{cases}k+\lambda(H-V_j) & \text{if } j=\ell,\\ \lambda H & \text{otherwise.}\end{cases} \tag{5}$$

for suitable subgroups $V_1,\cdots,V_t$ of H. Therefore $[D_{ij}^{(-1)}]^T$ is a $\mathrm{GH}(s,u,\lambda)$ matrix over H with respect to $V_j,\ 1\le j\le t$.

Conversely, assume that there exist subgroups $V_1,V_2,\cdots,V_t$ of H of order u satisfying (5). Then,

$$\begin{aligned}|B_{ja}\cap B_{\ell b}| &= \sum_{1\le i\le t}|D_{ij}a\cap D_{i\ell}b|\\ &= |\{(x,y): x\in D_{ij}, y\in D_{i\ell}, x^{-1}y=ab^{-1},\ 1\le i\le t\}|.\end{aligned}$$

Hence, if $a\neq b$, then by (5) we have

$$|B_{ja}\cap B_{\ell b}|=\begin{cases}0 & \text{if } j=\ell \text{ and } ab^{-1}\in V_j\setminus\{1\},\\ \lambda & \text{otherwise.}\end{cases}$$

It follows that $(\mathbb{B},\mathbb{P})$ is a dual transversal design $\mathrm{TD}_\lambda[k;u]$ with the block classes B_{j,V_jx} $(1\le j\le t, x\in H)$. Thus $(\mathbb{P},\mathbb{B})$ is symmetric. □

By Lemma 2.2 and Theorem 3.10, we have

Corollary 3.11. *Let $[D_{ij}]$ be a GH(s,u,λ) matrix over a group H with respect to a normal subgroup U of H. Then $[D_{ij}^{(-1)}]^T$ is also a GH(s,u,λ) matrix over a group H with respect to U.*

Example 3.12. Let $[D_{ij}]$ be the GH$(3,3,2)$ matrix in Exampme 3.6. Then $D_{11}^{(-1)}D_{11}+D_{21}^{(-1)}D_{21}=6+(H-\langle a\rangle)+(H-\langle b\rangle)$. Applying Theorem 3.10 the transversal design obtained from $[D_{ij}]$ is not symmetric.

Similarly, the transversal design obtained from a GH$(3,3,4)$ matrix in Example 3.7 is not symmetric as we can check that $D_{11}^{(-1)}D_{11}+D_{21}^{(-1)}D_{21}=12+2(ab^2+ab^4+a^2b^2+a^2b^4)+3(ab+a^2b^5)+4(a+a^2+b+b^2+b^3+b^4+b^5+ab^3+a^2b^3)+5(ab^5+a^2b)$.

4 Transversal designs constructed from spreads

In this section we construct transversal designs from spreads as an application of Theorem 3.5.

Let H be a group of order $q^2(>1)$ and $\{H_1,\cdots,H_{q+1}\}$ a spread of H. We show that many GH(q,q,q) matrices can be constructed from each spread.

Theorem 4.1. Let q be a power of a prime p and Let $\{H_1,\cdots,H_{q+1}\}$ be a spread of an elementary abelian p-group H of order q^2. Let $A=[n_{ij}]$ be a $q\times q$ matrix with entries from $I=\{1,2,\cdots,q+1\}$ satisfying the following.

$$I=\{n_{i1},n_{i2},\cdots,n_{iq},m_i\},\quad 1\le i\le q, \tag{6}$$

and

$$I=\{n_{1j},n_{2j},\cdots,n_{qj},\ell_j\},\quad 1\le j\le q, \tag{7}$$

for some $m_1,\cdots,m_q\in I$ and $\ell_1,\cdots,\ell_q\in I$. Set $D_{ij}=H_{n_{ij}}$ for each i,j with $1\le i,j\le q$. Then $[D_{ij}]$ is a GH(q,q,q) matrix with respect to $H_{m_1},\dots,H_{m_q}$ and the transversal design TD$_q[q^2;q]$ corresponding to $[D_{ij}]$ is symmetric.

Proof. Clearly, $H_{n_{i1}}H_{n_{i1}}^{(-1)}+H_{n_{i2}}H_{n_{i2}}^{(-1)}+\cdots+H_{n_{iq}}H_{n_{iq}}^{(-1)}=q(H_{n_{i1}}+\cdots+H_{n_{iq}})=q^2+q(H-H_{m_i})$ by Lemma 2.1(ii). Moreover, by Lemma 2.1(i), $H_{n_{i1}}H_{n_{j1}}^{(-1)}+H_{n_{i2}}H_{n_{j2}}^{(-1)}+\cdots+H_{n_{iq}}H_{n_{jq}}^{(-1)}=qH$ if $j\neq\ell$. Thus $[D_{ij}]$ is a GH(q,q,q) matrix with respect to $H_{m_1},\dots,H_{m_q}$. Cleary, similar conditions hold for columns of $[D_{ij}^{(-1)}]$. Hence $[D_{ij}^{(-1)}]^T$ is also a GH(q,q,q) matrix with respect to $H_{\ell_1},\dots,H_{\ell_q}$. Therefore the transversal design corresponding to $[D_{ij}]$ is symmetric applying Theorem 3.10. $\square$

Remark 4.2. Transversal designs obtained by Theorem 4.1 are always symmetric but do not always admit class regular automorphism groups. For example, let $q=3$ and $G=\langle a,b\rangle\simeq\mathbb{Z}_3\times\mathbb{Z}_3$ and set $M=\begin{bmatrix}\langle a\rangle & \langle ab\rangle & \langle a^2b\rangle\\ \langle ab\rangle & \langle a^2b\rangle & \langle a\rangle\\ \langle a^2b\rangle & \langle b\rangle & \langle ab\rangle\end{bmatrix}$. Then we can check by a computer search that the symmetric transversal design obtained from M does not admit a class regular autmorphism group. Therefore this is one of the two symmetric transversal designs obtained by V.C. Mavron and V.D. Tonchev (see Table 3 of [10]).

5 Product construction from GH(s, u, λ)'s

It is well known that from a GH(u, λ) matrix and a GH(u, λ') matrix over a group U one can construct a GH$(u, u\lambda\lambda')$ matrix by Kronecker product ([4]). In this section we generalize this method to a GH(s, u, λ) matrix and a GH(s', u, λ') matrix.

First we define a transformed Kronecker product in the following way.

Definition 5.1. Let G and N be groups. For each $i, 1 \le i \le n$, let f_i be a monomorphism from N into G. For $b = \sum_{x \in N} c_x x$ $(\in \mathbb{Z}[N])$ we define b^{f_i} by $b^{f_i} = \sum_{x \in N} c_x x^{f_i}$ $(\in \mathbb{Z}[G])$. For $A = [a_{ij}] \in \mathrm{M}_n(\mathbb{Z}[G])$ and $B = [b_{ij}] \in \mathrm{M}_r[\mathbb{Z}[N]]$, a nr by nr matrix $A \otimes B^{(f_1, \cdots, f_n)}$ labelled with $\{(i, j) : 1 \le i \le n,\ 1 \le j \le r\}$ is defined by

$$A \otimes B^{(f_1, \cdots, f_n)} = \begin{bmatrix} B^{f_1} a_{11} & B^{f_1} a_{12} & \cdots & B^{f_1} a_{1n} \\ B^{f_2} a_{21} & B^{f_2} a_{22} & \cdots & B^{f_2} a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ B^{f_n} a_{n1} & B^{f_n} a_{n2} & \cdots & B^{f_n} a_{nn} \end{bmatrix},$$
$$\textit{where} \quad B^{f_\ell} = [b_{ij}^{f_\ell}]$$

Using this product we can show the following.

Theorem 5.2. *Let G be a group. Let H be a normal subgroup of G of order su and H' a subgroup of G of order $s'u$ satisfying $G = HH'$ (the product operation on G), $U = H \cap H'$ and $|U| = u$. Let $D = [D_{ij}] \in \mathrm{M}_t(\mathbb{Z}[H])$ $(t = u\lambda/s)$ be a GH(s, u, λ) matrix over H with respect to U_i $(1 \le i \le t)$ and let $W = [W_{\ell m}] \in \mathrm{M}_{t'}(\mathbb{Z}[H'])$ $(t' = u\lambda'/s')$ be a GH(s', u, λ') matrix over H' with respect to U. Assume that there exists a monomorphism f_i from H' into G satisfying the following :*

$$U^{f_i} = U_i, \quad G = H(H')^{f_i} \ (\textit{the product operation on } G) \tag{8}$$

for each $i,\ 1 \le i \le t$. Set $\Delta = \{1, \cdots, t\} \times \{1, \cdots, t'\}$. Then we have

(i) *The tt' by tt' matrix $D \otimes W^{(f_1, \cdots, f_t)}$ labelled with Δ is a GH$(ss', u, u\lambda\lambda')$ matrix over G with respect to $U_{(i,\ell)} = U_i$ $((i, \ell) \in \Delta)$.*

(ii) *The transversal design corresponding to $D \otimes W^{(f_1, \cdots, f_t)}$ is symmetric if and only if the transversal design corresponding to D is symmetric.*

Proof. Set $k = u\lambda$, $k' = u\lambda'$ and $M = D \otimes W^{(f_1, \cdots, f_t)}$. Set $M = [M_{(i,\ell),(j,m)}]$ $((i,\ell),(j,m) \in \Delta)$. Then,

$$M_{(i,\ell),(j,m)} = W^{f_i}_{\ell m} D_{ij} \quad (1 \le i, j \le t,\ 1 \le \ell, m \le t'). \tag{9}$$

By assumption,

$$\sum_{j=1}^{t} D_{ij} D^{(-1)}_{\ell j} = \lambda H \quad (1 \le i \ne \ell \le t) \tag{10}$$

$$\sum_{j=1}^{t} D_{ij} D^{(-1)}_{ij} = k + \lambda(H - U_i) \quad (1 \le i \le t) \tag{11}$$

$$\sum_{j=1}^{t'} W_{ij} W^{(-1)}_{\ell j} = \lambda' H' \quad (1 \le i \ne \ell \le t') \tag{12}$$

$$\sum_{j=1}^{t'} W_{ij} W^{(-1)}_{ij} = k' + \lambda'(H' - U) \quad (1 \le i \le t') \tag{13}$$

We note that

$$(H')^{f_i} \rhd U^{f_i} = U_i \tag{14}$$

and if we regard H and $(H')^{f_i}$ as elements of $\mathbb{Z}[G]$, then $H(H')^{f_i} = uG$. We show that

$$\sum_{(m,p)\in\Delta} M_{(i,a),(m,p)} M^{(-1)}_{(j,b),(m,p)} = \begin{cases} kk' + \lambda\lambda' u(G - U_i) & \text{if } (i,a) = (j,b), \\ \lambda\lambda' uG & \text{otherwise.} \end{cases}$$

First assume that $(i,a) = (j,b)$. By (9) and (11),

$$\begin{aligned}
&\sum_{(m,p)\in\Delta} M_{(i,a),(m,p)} M^{(-1)}_{(i,a),(m,p)} = \sum_{(m,p)\in\Delta} W^{f_i}_{ap} D_{im} (W^{f_i}_{ap} D_{im})^{(-1)} \\
&= \sum_{1\le p\le t'} \sum_{1\le m\le t} W^{f_i}_{ap} (D_{im} D^{(-1)}_{im}) (W^{(-1)}_{ap})^{f_i} \\
&= \sum_{1\le p\le k'} W^{f_i}_{ap} (k + \lambda(H - U_i)) (W^{(-1)}_{ap})^{f_i} \\
&= k \sum_{1\le p\le t'} (W_{ap} W^{(-1)}_{ap})^{f_i} + \lambda \sum_{1\le p\le t'} W^{f_i}_{ap} H (W^{(-1)}_{ap})^{f_i} \\
&\qquad - \lambda \sum_{1\le p\le t'} W^{f_i}_{ap} U_i (W^{(-1)}_{ap})^{f_i}.
\end{aligned} \tag{15}$$

Since $G \rhd H$ and $H' \rhd U$, it follows from (13), (14) and (15) that

$$\begin{aligned}
&\sum_{(m,p)\in\Delta} M_{(i,a),(m,p)} M^{(-1)}_{(i,a),(m,p)} \\
&= k(k' + \lambda'(H' - U))^{f_i} + \lambda H(k' + \lambda' H' - \lambda' U)^{f_i} - \lambda U_i(k' + \lambda'(H' - U))^{f_i} \\
&= kk' + k\lambda'(H' - U)^{f_i} + \lambda H(k' + \lambda'(H')^{f_i} - \lambda' U_i) - \lambda k' U_i - \lambda\lambda' U^{f_i}(H' - U)^{f_i} \\
&= kk' + \lambda(k'H + \lambda' uG - \lambda' uH) - \lambda k' U_i = kk' + \lambda\lambda' u(G - U_i).
\end{aligned}$$

Assume $i = j$ and $a \neq b$. By (11) and (12),

$$\begin{aligned}
&\sum_{(m,p)\in\Delta} M_{(i,a),(m,p)} M^{(-1)}_{(i,b),(m,p)} = \sum_{1\le p\le t'} \sum_{1\le m\le t} W^{f_i}_{ap}(D_{im} D^{(-1)}_{im})(W^{(-1)}_{bp})^{f_i} \\
&= \sum_{1\le p\le t'} W^{f_i}_{ap}(k + \lambda(H - U_i))(W^{(-1)}_{bp})^{f_i} \\
&= k \sum_{1\le p\le t'} (W_{ap} W^{(-1)}_{bp})^{f_i} + \lambda \sum_{1\le p\le t'} H(W_{ap} W^{(-1)}_{bp})^{f_i} - \lambda \sum_{1\le p\le t'} U^{f_i} W^{f_i}_{ap}(W^{(-1)}_{bp})^{f_i} \\
&= u\lambda\lambda'(H')^{f_i} + \lambda H(\lambda' H')^{f_i} - \lambda U^{f_i}(\lambda' H')^{f_i} = \lambda\lambda' uG.
\end{aligned}$$

Assume $i \neq j$. We note that $|W^{f_i}_{ap}| = [G : H] = s'$. If $x^{f_i} \neq y^{f_i} \in W^{f_i}_{ap}$, then $x^{f_i}(y^{f_i})^{(-1)} \notin U^{f_i} = U_i$. On the other hand $(H')^{f_i} \cap H = U_i$. Hence $W^{f_i}_{ap}$ is a complete set of coset representatives of G/H. Consequently, $W^{f_i}_{ap} H = G$.
By (10),

$$\begin{aligned}
\sum_{(m,p)\in\Delta} M_{(i,a),(m,p)} M^{(-1)}_{(j,b),(m,p)} &= \sum_{1\le p\le t'} \sum_{1\le m\le t} W^{f_i}_{ap} D_{im} D^{(-1)}_{jm}(W^{(-1)}_{bp})^{f_j} \\
&= \sum_{1\le p\le t'} W^{f_i}_{ap}(\lambda H)(W^{(-1)}_{bp})^{f_j} \\
&= \lambda \sum_{1\le p\le t'} H W^{f_i}_{ap}(W^{(-1)}_{bp})^{f_j} \\
&= \lambda \sum_{1\le p\le t'} G(W^{(-1)}_{bp})^{f_j} = \lambda t' s' G = \lambda\lambda' uG.
\end{aligned}$$

Thus $[\, M_{(i,\ell),(j,m)}\,]$ is a $\mathrm{GH}(ss', u, u\lambda\lambda')$ matrix over G and so (i) holds.

To prove (ii) we consider $(M^{(-1)})^T$.

$$\begin{aligned}
&\sum_{(i,\ell)\in\Delta} M^{(-1)}_{(i,\ell),(j_1,m_1)} M_{(i,\ell),(j_2,m_2)} \\
&= \sum_{(i,\ell)\in\Delta} (W^{f_i}_{\ell m_1} D_{ij_1})^{(-1)}(W^{f_i}_{\ell m_2} D_{ij_2}) \\
&= \sum_{1\le i\le t} \sum_{1\le \ell\le t'} D^{(-1)}_{ij_1}(W^{(-1)}_{\ell m_1} W_{\ell m_2})^{f_i} D_{ij_2}. \qquad (16)
\end{aligned}$$

By Corollary 3.11,

$$\sum_{1\le i\le t'} W_{i,m_1}^{(-1)} W_{i,m_2} = \begin{cases} k' + \lambda'(H' - U) & \text{if } m_1 = m_2, \\ \lambda' H' & \text{otherwise.} \end{cases} \tag{17}$$

By (16) and (17),

$$\begin{aligned} &\sum_{(i,\ell)\in\Delta} M_{(i,\ell),(j_1,m_1)}^{(-1)} M_{(i,\ell),(j_2,m_2)} \\ &= \begin{cases} \sum_{1\le i\le t} D_{ij_1}^{(-1)} (k' + \lambda'(H' - U)^{f_i}) D_{ij_2} & \text{if } m_1 = m_2, \\ \sum_{1\le i\le t} D_{ij_1}^{(-1)} (\lambda' H')^{f_i} D_{ij_2} & \text{otherwise.} \end{cases} \end{aligned} \tag{18}$$

As $[H : U_i] = |D_{ij}| = s$, by (11) D_{ij} is a complete set of left coset representatives of H/U_i. Hence $U_i D_{ij} = H$. By assumption, $|G| = |H| \cdot |H'|/|H \cap H'|$ and $|G| = |H| \cdot |(H')^{f_i}|/|H \cap (H')^{f_i}|$. Hence $|H \cap (H')^{f_i}| = |H \cap H'| = |U| = u$. It follows that $[G : (H')^{f_i}] = s$ and $(H')^{f_i} \cap H = U_i$. Thus, if $d_1 d_2^{-1} \in (H')^{f_i}$ for some $d_1, d_2 \in D_{ij}$, then $d_1 d_2^{-1} \in (H')^{f_i} \cap H = U_i$, which forces $d_1 = d_2$. Therefore D_{ij} is a complete set of left coset representatives of $G/(H')^{f_i}$. In particular, $(H')^{f_i} D_{ij} = G$. It follows from (18) that

$$\begin{aligned} &\sum_{(i,\ell)\in\Delta} M_{(i,\ell),(j_1,m_1)}^{(-1)} M_{(i,\ell),(j_2,m_2)} \\ &= \begin{cases} k' \sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_2} + u\lambda\lambda' G - u\lambda\lambda' H & \text{if } m_1 = m_2, \\ u\lambda\lambda' G & \text{otherwise.} \end{cases} \end{aligned} \tag{19}$$

By Theorem 3.10, the necessary and sufficient condition for the transversal design obtained from $[M_{(i,\ell),(j,m)}]$ to be symmetric is that there exist subgroups $V_{(j,m)}$, $(j,m) \in \Delta$, of G of order u satisfying

$$\begin{aligned} &\sum_{(i,\ell)\in\Gamma} M_{(i,\ell),(j_1,m_1)}^{(-1)} M_{(i,\ell),(j_2,m_2)} \\ &= \begin{cases} kk' + u\lambda\lambda'(G - V_{(j_1,m_1)}) & \text{if } (j_1,m_1) = (j_2,m_2), \\ u\lambda\lambda' G & \text{otherwise.} \end{cases} \end{aligned} \tag{20}$$

Assume that the transversal design obtained from $D \otimes W^{(f_1,\cdots,f_t)}$ is symmetric. We compare (19) with (20). If $m_1 = m_2$ and $j_1 = j_2$, then $kk' + u\lambda\lambda'(G - V_{(j_1,m_1)}) = k' \sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_1} + u\lambda\lambda' G - u\lambda\lambda' H$. Hence $\sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_1} = k + \lambda(H - V_{(j_1,m_1)})$. If $m_1 = m_2$ and $j_1 \ne j_2$, then $k' \sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_2} + u\lambda\lambda' G - u\lambda\lambda' H = u\lambda\lambda' G$. Hence $\sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_2} = \lambda H$. By Theorem 3.10 the transversal design obtained from D is symmetric.

Conversely, assume that the transversal design obtained from D is symmetric. Then, by Theorem 3.10 we have

$$\sum_{1\le i\le t} D_{ij_1}^{(-1)} D_{ij_2} = \begin{cases} k + \lambda(H - V_{j_1}) & \text{if } j_1 = j_2, \\ \lambda H & \text{otherwise.} \end{cases}$$

for some subgroups $V_j,\ 1 \le j \le t$ of H of order u. Thus, by (19)

$$\sum_{(i,\ell)\in\Gamma} M^{(-1)}_{(i,\ell),(j_1,m_1)} M_{(i,\ell),(j_2,m_2)}$$

$$= \begin{cases} k'(k+\lambda(H-V_{j_1})) + u\lambda\lambda' G - u\lambda\lambda' H & \text{if } m_1 = m_2 \text{ and } j_1 = j_2, \\ k'\lambda H + u\lambda\lambda' G - u\lambda\lambda' H & \text{if } m_1 = m_2 \text{ and } j_1 \ne j_2, \\ u\lambda'\lambda G & \text{if } m_1 \ne m_2. \end{cases}$$

$$= \begin{cases} kk' + u\lambda\lambda'(G - V_{j1}) & \text{if } m_1 = m_2 \text{ and } j_1 = j_2, \\ u\lambda'\lambda G & \text{otherwise.} \end{cases}$$

Therefore, by Theorem 3.10, the transversal design obtained from $D \otimes W^{(f_1,\cdots,f_t)}$ is symmetric. □

Example 5.3. Let $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a matrix over $\mathbb{Z}_3$. Then W is a generalized Hadamard matrix GH$(3,1)$ over $\mathbb{Z}_3$. Let $D = [D_{ij}]$ be a GH$(3,3,2)$ matrix over $H = \langle a,b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ in Example 3.6. We define monomorphisms f_1 and f_2 from $\mathbb{Z}_3$ to H by $f_1(x) = b^x$ and $f_2(x) = a^x$ ($x \in \mathbb{Z}_3$), respectively. Then, applying Theorem 5.2 repeatedly, $D \otimes (\otimes_{i=1}^n W)^{(f_1,f_2)}$ is a GH$(3,3,2\cdot 3^n)$ matrix over H and the corresponding transversal design is non-symmetric.

Similarly, let $D = [D_{ij}]$ be a GH$(3,3,4)$ matrix over $H = \langle a,b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ in Example 3.7. Using monomorphisms f_i ($1 \le i \le 4$) from $\mathbb{Z}_3$ to H defined by $f_i(x) = a^x$ ($i = 1,3$) and $f_i(x) = b^x$ ($i = 2,4$) for $x \in \mathbb{Z}_3$, we obtain a GH$(3,3,4\cdot 3^n)$ matrix $D \otimes (\otimes_{i=1}^n W)^{(f_1,\cdots,f_4)}$ over H. The corresponding transvesal design $\mathrm{TD}_{3^n 4}[4\cdot 3^{n+1};3]$ is non-symmetric.

As corollaries of Theorem 5.2, we have

Corollary 5.4. *Let H_1 and H_2 be normal subgroups of G satisfying $G = H_1H_2$ (the product operation on G), $U = H_1 \cap H_2$, $|U| = u$. If $[C_{ij}] \in M_{t_1}(\mathbb{Z}[H_1])$ is a GH(s_1,u,λ_1) matrix, $t_1 = \frac{u\lambda}{s_1}$, over H_1 with respect to U and if $[D_{ij}] \in M_{t_2}(\mathbb{Z}[H_2])$ is a GH(s_2,u,λ_2) matrix, $t_2 = \frac{u\lambda_2}{s_2}$, over H_2 with respect to U, then $C \otimes D^{\mathrm{id}_U} (= [C_{i,j}D_{p,q}]\ ((i,p),(j,q) \in \{1,\cdots,t_1\} \times \{1,\cdots,t_2\})$ is a GH$(s_1s_2,u,\lambda_1\lambda_2 u)$ matrix over H, where id_U denotes the identity map.*

Corollary 5.5. *Let $D = [D_{ij}]$ be a t by t GH(s,u,λ) matrix over a group H with respect to subgroups U_i $(1 \le i \le t)$ of H and let $W = [w_{\ell m}]$ be a generalized Hadamard matrix GH(u,λ') over a group U. If there exists monomorphism f_i from U to U_i for each i with $1 \le i \le t$, then $D \otimes W^{(f_1,\cdots,f_t)}$ is a GH$(s,u,u\lambda\lambda')$ matrix over H. Moreover, the transversal design obtained from $D \otimes W^{(f_1,\cdots,f_t)}$ is symmetric if and only if the transversal design obtained from D is symmetric.*

Remark 5.6. Corollary 5.4 is a generalization of Davis' product construction of semiregular relative difference sets ([3]).

6 Construction of GH(s,u,λ) in a subgroup

In this section for a given GH(s,u,λ) matrix over H we construct a GH(s_1,u,λ) matrix over some suitable subgroup N of H of order s_1u with $s_1 \mid s$.

For any subset $Y(\neq \emptyset)$ of a group G, Y^G denotes the subgroup of G generated by the sets $g^{-1}Yg$ with $ginG$. Y^G is called *the normal closure* of Y in G.

Proposition 6.1. *Let $[D_{ij}]$ be a t by t GH(s,u,λ) matrix over a group H of order su with respect to subgroups U_i $(1 \le i \le t)$ of H of order u. Let N be a subgroup of H satisfying $N \ge \langle {U_1}^H, \cdots, {U_t}^H\rangle$. Set $|N| = s_1u$ and $r = [H : N]$ and choose a complete set of right coset representatives $g_1, \cdots, g_r$ with respect to N :*

$$H = g_1N \cup g_2N \cup \cdots \cup g_rN \tag{21}$$

Define an rt by rt matrix $[C_{(i,\ell),(j,m)}]$ $(1 \le i,j \le t,\ 1 \le \ell, m \le r)$ by $C_{(i,\ell),(j,m)} = N \cap g_\ell^{-1}D_{ij}g_m$. Then $[C_{(i,\ell),(j,m)}]$ is a GH(s_1,u,λ) matrix over N with respect to $U_{(i,\ell)} = {g_\ell}^{-1}U_ig_\ell$ $(1 \le i \le t, 1 \le \ell \le r)$.

Proof. Let $\mathcal{D} = (\mathbb{P},\mathbb{B})$ be a transversal design TD$_k[u,\lambda]$ defined in Theorem 3.5. Our aim is to apply Theorem 3.2 to the transversal design $\mathcal{D}$ with respect to the subgroup N. By (iv) of Theorem 3.5, each (i, U_ig) $(1 \le i \le t, g \in H)$ is a point class of $\mathcal{D}$. As ${U_i}^H \le N$, $U_ig_\ell N = g_\ell({g_\ell}^{-1}U_ig_\ell)N = g_\ell N$. Hence, by (21), each N-orbit on $\mathbb{P}$ is of the form $(i, g_\ell N)$ and it contains the point class (i, U_ig_ℓ). We choose a point (i, g_ℓ) $(1 \le i \le t, 1 \le \ell \le r)$ on the N-orbit $(i, g_\ell N)$. Let $U_{(i,\ell)}$ the subgroup corresponding to $(i, g_\ell N)$. Then $U_{(i,\ell)} = \{x \in N : (i,g_\ell)x \sim (i,g_\ell)\} = \{x \in N : (i, g_\ell x) \in (i, U_ig_\ell)\} = {g_\ell}^{-1}U_ig_\ell$.

On the other hand, we can choose a block B_{j,g_m} $(1 \le j \le t, 1 \le m \le r)$ on each N-orbit $(\subset \mathbb{B})$. Since $B_{j,g_m} = \sum_{w=1}^{t}(w, D_{wj}g_m)$, it follows that $\{x \in N : (i,g_\ell)x \in B_{j,g_m}\} = \{x \in N : g_\ell x \in D_{ij}g_m\} = N \cap g_\ell^{-1}D_{ij}g_m = C_{(i,\ell),(j,m)}$ $(1 \le i,j \le t,\ \ 1 \le \ell, m \le r)$. By Theorem 3.2, we have the proposition. $\square$

Example 6.2. Let $[D_{ij}]$ be a GH$(6,3,4)$ matrix over $H = \langle a,b\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_6$ in Example 3.7. Set $N = \langle U_1, U_2\rangle$. Then $N = \langle a, b^2\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Then we have a right coset decomposition $H = 1N + bN$ with respect to N. Hence, by Proposition 6.1, we have a GH(3,3,4) matrix $C = [C_{ij}] \in \mathrm{M}_4(\mathbb{Z}[N])$, where

$$C = \begin{bmatrix} 1+b^2+a & b^2+b^4+ab^2 & 1+ab^4+b^4 & a^2+a^2b^2+ab^2 \\ 1+b^2+a & 1+b^2+a & a^2b^4+a^2+a & 1+ab^4+b^4 \\ 1+a^2b^4+b^4 & a+ab^2+a^2b^2 & 1+b^2+a^2 & b^2+b^4+a^2b^2 \\ ab^4+a+a^2 & 1+a^2b^4+b^4 & 1+b^2+a^2 & 1+b^2+a^2 \end{bmatrix}.$$

By Proposition 6.1 we have the following.

Proposition 6.3. *Let $[D_{ij}]$ be a t by t GH(s,u,λ) matrix over a group H of order su with respect to a normal subgroup U of H. Let $g_1, \cdots, g_s$ be a complete set of coset representatives of H/U. Set $C_{(i,\ell),(j,m)} = U \cap g_\ell^{-1}D_{ij}g_m$. Then the $u\lambda$ by $u\lambda$ matrix $[C_{(i,\ell),(j,m)}]$ is a generalized Hadamard matrix GH(u,λ) over U.*

Applying Proposition 6.3 to semiregular relative difference sets we have the following.

Proposition 6.4. *Let R be a $(u\lambda, u, u\lambda, \lambda)$-difference set in a group G relative to a subgroup U. We choose any subgroup H containing the normal closure U^G of U in G. Set $|H| = us$ and $st = u\lambda$. Let $G = g_1H \cap \cdots \cap g_tH$ be a right coset decomposition with respect to H and set $D_{ij} = H \cap g_i^{-1}Rg_j$ and $U_i = U^{g_i}$ $(1 \leq i, j \leq t)$. Then $[D_{ij}]$ is the $GH(s, u, \lambda)$ matrix over H with respect to U_i, $1 \leq i \leq t$.*

Proof. Since $[R]$ is a 1 by 1 GH$(u\lambda, u, \lambda)$ matrix with respect to U, the proposition immediately follows from Proposition 6.1. □

Remark 6.5. In Proposition 6.4, $(\mathbb{P}, \mathbb{B}) = (G, \{Dh : h \in G\})$ by (2) and each point class of $(\mathbb{P}, \mathbb{B})$ is given by Ux on which a subgroup $x^{-1}Ux$ acts regularly. Moreover, by Theorem 3.10, $(\mathbb{P}, \mathbb{B})$ is symmetric if and only if $D^{(-1)}D = u\lambda + \lambda(G - V)$ for some subgroup V of G of order u (cf. Proposition 2.6 of [5]).

References

[1] K. Akiyama, M. Ogawa and C. Suetake, On STD$_6$[18; 3]'s and STD$_7$[21; 3]'s admitting a semiregular automorphism group of order 9, preprint.

[2] T. Beth, D. Jungnickel and H. Lenz, "Design Theory" Volume I, Second Edition, Cambridge University Press, 1999.

[3] J. A. Davis, A Note on Products of Relative Difference Sets, Designs, Codes and Cryptography, Vol. 1 (1991), 117-119.

[4] D. A. Drake, Partial λ-geometries and generalized Hadamard matrices over groups, Canad. J. Math. Vol. 31 (1979), 617-627.

[5] Y. Hiramine, On non-symmetric relative difference sets, Hokkaido Math. J. Vol. 37 (2008), 427-435.

[6] D. R. Hughes and F. C. Piper, "Projective Planes", Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[7] D. Jungnickel, On automorphism groups of divisible designs, Canad. J. Math. Vol. 34, 1982, 257–297.

[8] H. Lüneburg, "Translation Planes", Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[9] A. Pott, "Finite Geometry and Character Theory", Springer-Verlag, New York-Heidelberg-Berlin, 1995.

[10] V.C. Mavron and V.D. Tonchev,On symmetric nets and generalized Hadamard matrices from affine designs, J. of Geometry, Vol. 67, 2000, 180-187.