BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY FOR DIVISORS ON RATIONAL NORMAL SCROLLS

CHIKASHI MIYAZAKI

ABSTRACT. The Castelnuovo-Mumford regularity is one of the most important invariants in studying the minimal free resolution of the defining ideals of the projective varieties. There are some bounds on the Castelnuovo-Mumford regularity of the projective variety in terms of the other basic invariants such as dimension, codimension and degree. This paper studies a bound on the regularity conjectured by Hoa, and shows this bound and extremal examples in the case of divisors on rational normal scrolls.

1. INTRODUCTION

Let X be a projective scheme of $\mathbb{P}_K^N$ over an algebraic closed field K. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Then we put $\mathbb{P}_K^N = \mathrm{Proj}(S)$. We denote by $\mathcal{I}_X$ the ideal sheaf of X. Let m be an integer. Then X is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_K^N, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_K^N$, introduced by Mumford by generalizing ideas of Castelnuovo, is the least such integer m and is denoted by $\mathrm{reg}(X)$. The interest in this concept stems partly from the well-known fact that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the p^{th} syzygy module of the defining ideal I of $X \subseteq \mathbb{P}_K^N$ occur in degree $\leq m + p$, see, e.g., [4]. It is important to study upper bounds on the Castelnuovo-Mumford regularity for projective schemes in order to describe the minimal free resolutions of the defining ideals.

In what follows, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to ℓ, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to ℓ.

The starting point of our research on the Castelnuovo-Mumford regularity is an inequality $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ for the ACM, that is, arithmetically Cohen-Macaulay, nondegenerate projective variety $X \subseteq \mathbb{P}_K^N$, which is a consequence of the Uniform Position Lemma for the generic hyperplane section of the projective curve for the characteristic zero case and

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the corresponding weaker result due to Ballico for the positive characteristic case, see [1, 2]. Moreover, the extremal ACM variety for the bound have been shown to be a variety of minimal degree in [14, 18] if its degree is large enough.

In order to study the regularity bounds for the non-ACM projective variety, we introduce the k-Buchsbaum property. Let k be a nonnegative integer. Then X is called k-Buchsbaum if the graded S-module $\mathrm{M}^i(X) = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(\ell))$, which is called the deficiency module or the Hartshorne-Rao module of X, is annihilated by $\mathfrak{m}^k$ for $1 \le i \le \dim(X)$, see, e.g., [9, 10]. We call the minimal nonnegative integer n, if it exists, such that X is n-Buchsbaum, as the Ellia-Migliore-Miró Roig number of X and denote it by $k(X)$, see [3, 12]. Further we define $\tilde{k}(X)$ as the maximal integer k such that all successive hyperplane sections of X, that is, $X \cap L$ with $\operatorname{codim}(X \cap L) = \operatorname{codim}(X) + \operatorname{codim}(L)$ for any linear space L of $\mathbb{P}^N_K$, have the k-Buchsbaum property, see [5]. Note that $k(X) < \infty$ if and only if $\tilde{k}(X) < \infty$, which is equivalent to saying that X is locally Cohen-Macaulay and equi-dimensional. In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety X have been given by several authors in terms of $\dim(X)$, $\deg(X)$, $\operatorname{codim}(X)$ and $k(X)$, see, e.g., [6, 7, 13, 16]. The following bound is the most optimal among the known results. Also, the extremal cases are classified, see, e.g., [3, 12].

Proposition 1.1 (See [3, 16])**.** *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K. Then we have* $\operatorname{reg}(X) \le \lceil(\deg(X) - 1)/\operatorname{codim}(X)\rceil + \max\{k(X)\dim(X), 1\}$*. Assume that X is not ACM and that* $\deg(X) \ge 2\operatorname{codim}(X)^2 + \operatorname{codim}(X) + 2$*. Then the equality holds only if X is a curve on a rational ruled surface.*

This motivates us to state a variation of Hoa's conjecture.

Conjecture 1.2 ([12])**.** *Let X be a nondegenerate projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K. Then we have* $\operatorname{reg}(X) \le \lceil(\deg(X) - 1)/\operatorname{codim}(X)\rceil + \max\{\tilde{k}(X), 1\}$*. Furthermore, assume that X is not ACM and that* $\deg(X)$ *is large enough. Then the equality holds only if X is a divisor on a rational normal scroll.*

We remark that the original Hoa's conjecture takes $\bar{k}(X)$ instead of $\tilde{k}(X)$, where $\bar{k}(X)$ is the maximal integer k such that all successive *hypersurface* sections of X have the k-Buchsbaum property. The Buchsbaum case, that is, $\tilde{k}(X) = 1$, has been proved in [15, 17, 19].

The purpose of this paper is to prove the conjecture for divisors on rational normal scrolls and to give extremal varieties for all dimensions.

Theorem 1.3. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ of dimension r over an algebraically closed field K. Put $k = \tilde{k}(X)$. Assume that X is a divisor on a rational normal scroll. Then we have* $\operatorname{reg}(X) \le \lceil(\deg(X) - 1)/\operatorname{codim}(X)\rceil + \max\{k, 1\}$*. Furthermore, there exist extremal examples for all r and k.*

Before proving the inequality and describing the extremal cases for divisors on rational normal scrolls, we prepare the following notations. Let $r \geq 2$ be an integer. Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_K$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_r)$ for some $0 \leq e_1 \leq \cdots \leq e_r$. Let Z and F be a minimal section and a fibre respectively. Now we have an embedding of Y in $\mathbb{P}^N_K$ by a very ample divisor $H = Z + nF$ $(n > e_r)$, where $N = rn + r + n - e_1 - \cdots - e_r$. Then Y is called a rational normal scroll. Let X be a divisor on Y linearly equivalent to $aZ + bF$. If X is nondegenerate, then $\Gamma(Y, \mathcal{I}_{X/Y}(1)) = \Gamma(Y, \mathcal{O}_Y((1-a)Z + (n-b)F)) = 0$. In this case we see that either $a = 1$ and $b \geq n+1$, or $a \geq 2$ and $b \geq 1$. Also, we have $\operatorname{codim}(X) = rn + n - e_1 - \cdots - e_r$ and $\deg(X) = (aZ + bF) \cdot (Z + nF)^r = a(rn - e_1 - \cdots - e_r) + b$, because $Z^{r+1} = -e_1 - \cdots - e_r$, $Z^r \cdot F = 1$ and $Z^i \cdot F^{r+1-i} = 0$ for $0 \leq i \leq r-1$. Under the above conditions, we obtain the following classification of the divisor on a rational normal scroll with its Castelnuovo-Mumford regularity having such upper bound.

Theorem 1.4. *Let X be a nondegenerate irreducible reduced divisor on a rational normal scroll in $\mathbb{P}^N_K$ of dimension r constructed as above. Then we have* $\operatorname{reg}(X) \leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + \max\{\tilde{k}(X), 1\}$.

Furthermore, assume that X is not ACM. Then the equality holds if and only if $a \geq 1$ and $an + 2 \leq b \leq an + 1 - (r+1)n - e_1 - \cdots - e_r$.

This result extends that of [12, Theorem 1.3] and give sharp examples for the conjecture. More precisely, the extremal variety X satisfies $\operatorname{codim}(X) = (r+1)n - e_1 - \cdots - e_r$, $\deg(X) = a(rn - e_1 - \cdots - e_r) + b$, $\tilde{k}(X) = k(X) = \lfloor (b - a_r - 2)/(n - e_r) \rfloor - a + 1$ and $\operatorname{reg}(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor + 2$.

2. Proof of Main Theorem

This section is devoted to the proof of the theorem stated in §1.

Notations being as in (1.4), our proof starts with calculating the Castelnuovo-Mumford regularity and the Ellia-Migliore-Miró Roig number of the projective variety. Let S be the polynomial ring $\Gamma(Y, \mathcal{O}_Y(1))$. Note that $\Gamma(Y, \mathcal{O}_Y(1)) \cong \Gamma(\mathbb{P}^1_K, \mathcal{E}(n))$. Since Y is ACM, the deficiency module $\mathrm{M}^i(X)$ of X in $\mathbb{P}^N_K = \operatorname{Proj}(S)$, $1 \leq i \leq r$, is isomorphic to $\oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(Y, \mathcal{I}_{X/Y}(\ell))$ as graded S-modules. Thus we have $\mathrm{M}^i(X) \cong \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(Y, \mathcal{O}_Y((-a+\ell)Z + (-b+\ell n)F))$ for $1 \leq i \leq r$. Let us calculate the intermediate cohomologies.

Lemma 2.1. *Under the above condition, we have*

(i) $\mathrm{M}^1(X) \cong \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^1(\mathbb{P}^1_K, \mathrm{Sym}^{\ell - a}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1_K}(n\ell - b))$,

(ii) $\mathrm{M}^i(X) = 0$ *for* $1 < i < r$, *and*

(iii) $\mathrm{M}^r(X) \cong \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^0(\mathbb{P}^1_K, (\mathrm{Sym}^{-\ell + a - r - 1}(\mathcal{E}))' \otimes \mathcal{O}_{\mathbb{P}^1_K}(n\ell - b + e_1 + \cdots + e_r))$.

Proof. The assertions immediately follow from [12, (2.13) and (2.14)] and their proofs. □

Corollary 2.2. *Under the above condition, we have*

(i) $\mathrm{M}^1(X)_\ell \neq 0$ *if and only if* $a \leq \ell \leq \lfloor (b - ae_r - 2)/(n - e_r) \rfloor$. *In particular,* $\mathrm{M}^1(X) \neq 0$ *if and only if* $b \geq an + 2$. *Furthermore,*

(ii) $\mathrm{M}^r(X)_\ell \neq 0$ *if and only if* $\lceil (b - e_1 - \cdots - e_{r-1} + (r-a)e_r)/(n - e_r) \rceil \leq \ell \leq a - r - 1$. *In particular,* $\mathrm{M}^r(X) \neq 0$ *if and only if* $b \leq (a - r - 1)n + e_1 + \cdots + e_r$.

Proof. Note that $n > e_r$. By (2.1)(i), $\mathrm{M}^1(X)_\ell \neq 0$ if and only if $-a + \ell \geq 0$ and $-b + \ell n \leq e_r(-a + \ell) - 2$. By (2.1)(iii), $\mathrm{M}^r(X)_\ell \neq 0$ if and only if $-a + \ell \leq -r - 1$ and $-b + \ell n \geq e_r(-a + \ell) + re_r - e_1 - \cdots - e_{r-1}$. □

Remark 2.3. From (2.2), X is ACM if and only if $(a - r - 1)n + e_1 + \cdots + e_r + 1 \leq b \leq an + 1$. If $b \geq an + 2$, then $\mathrm{M}^j(X) = 0$ for $j \neq 1$, and if $b \leq (a - r - 1)n + e_1 + \cdots + e_r$, then $\mathrm{M}^j(X) = 0$ for $j \neq r$. But both cases are not ACM.

Lemma 2.4. *Under the above condition,* $\mathrm{H}^{r+1}(\mathbb{P}^N_K, \mathcal{I}_X(\ell)) \neq 0$ *if and only if* $\ell \leq a - r - 1$ *and* $\ell \leq \lfloor (b - 2 - e_1 - \cdots - e_r)/n \rfloor$.

Proof. From the short exact sequence $0 \to H^{r+1}_*(\mathcal{I}_X) \to H^{r+1}_*(\mathcal{I}_{X/Y}) \to H^{r+2}_*(\mathcal{I}_Y) \to 0$, we see that $H^{r+1}_*(\mathcal{I}_X)$ is the kernel of the homomorphism $H^1(\mathbb{P}^1_K, \mathrm{Sym}^{-\ell+a-r-1}(\mathcal{E})' \otimes \mathcal{O}_{\mathbb{P}^1_K}(n\ell - b + e_1 + \cdots + e_r)) \to H^1(\mathbb{P}^1_K, \mathrm{Sym}^{-\ell-r-1}(\mathcal{E})' \otimes \mathcal{O}_{\mathbb{P}^1_K}(n\ell + e_1 + \cdots + e_r))$. Thus $\mathrm{H}^{r+1}(\mathcal{I}_X(\ell)) \neq 0$ if and only if $-\ell + a - r - 1 \geq 0$ and $n\ell - b + e_1 + \cdots + e_r \leq -2$. □

Remark 2.5. The a-invariant of the coordinate ring R of X is defined as $a(R) = \max\{\ell \mid [\mathrm{H}^{\dim R}_{R_+}(R)]_\ell \neq 0\}$. Note that $\mathrm{H}^{r+1}_{R_+}(R) \cong \mathrm{H}^{r+1}_*(\mathbb{P}^N_K, \mathcal{I}_X)$. Therefore we have $a(R) = \min\{a - r - 1, \lfloor (b - 2 - e_1 - \cdots - e_r)/n \rfloor\}$.

From now on, we assume that X is not ACM.

Corollary 2.6. *Under the above conditions,* $k(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor - a + 1$ *and* $\mathrm{reg}(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor + 2$ *if* $b \geq an + 2$, *and* $k(X) = a - r - 1 - \lceil (b - e_1 - \cdots - e_{r-1} + (r-1)e_r)/(n - e_r) \rceil + 1$ *and* $\mathrm{reg}(X) = a, a + 1$ *if* $b \leq (a - r - 1)n + e_1 + \cdots + e_r$.

Proof. It immediately follows from (2.1), (2.2), (2.3) and (2.4). □

Lemma 2.7. *Under the above conditions, we have* $\tilde{k}(X) = k(X)$.

Proof. It immediately follows from [8, (2.4)] and (2.1). □

Before proving the main theorem, we state a basic fact on the regularity bound.

Proposition 2.8 ([16]). *Let X be a nondegenerate projective variety of dimension r with the coordinate ring R. Let s be a fixed integer with $1 \leq s \leq r$. Assume that X is not ACM and that the deficiency module $\mathrm{M}^i(X)$ vanishes for any $i \neq s$. Then we have* $\mathrm{reg}(X) \leq a(R/hR) + r + 1 + k(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + k(X)$, *where h is a general linear form of R*

Proof of Theorem 1.4. The inequality $\mathrm{reg}(X) \leq \lceil(\deg(X) - 1)/\mathrm{codim}(X)\rceil + \tilde{k}(X)$ follows straightforward from (2.3), (2.7) and (2.8).

First, in order to describe when the equality holds, we consider the case $b \leq (a-r-1)n + e_1 + \cdots + e_r$. In this case, the intermediate cohomologies appear only in $\mathrm{M}^r(X)$, and we note that $\max\{\ell \mid [\mathrm{M}^r(X)]_\ell \neq 0\} = a - r - 1$ by (2.2). Also, we see that $a(R) \leq a-r-1$ by (2.5). If $a(R) = a-r-1$, then $\mathrm{reg}(X) = (a-r-1)+1+r+1 = a+1$ and $a(R/hR) = a-r$. If $a(R) < a-r-1$, then $\mathrm{reg}(X) = (a-r-1)+1+r = a$ and $a(R/hR) = a-r-1$. In fact, by the structure of $\mathrm{M}^r(X)$, see (2.1), we have $[\mathrm{M}^r(X)/h\mathrm{M}^r(X)]_{a-r-1} \neq 0$. In any case, we have $\mathrm{reg}(X) = a(R/hR) + r + 1 \leq \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + 1 \leq \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + \tilde{k}(X)$, and the equality holds only if $\tilde{k}(X) = 1$, which is the Buchsbaum case and is classified by [15].

Next, for the case $b \geq an + 2$, we see that $\mathrm{reg}(X) = \lfloor(b - ae_r - 2)/(n - e_r)\rfloor + 2$ and $\tilde{k}(X) = \lfloor(b-ae_r-2)/(n-e_r)\rfloor - a + 1$ by (2.6) and (2.7). Thus the equality holds if and only if $\lceil(a(rn-e_1-\cdots-e_r)+b-1)/((r+1)n-e_1-\cdots-e_r)\rceil = a+1$, which is equivalent to saying that $-(rn+n-e_1-\cdots-e_r)+1 \leq -na-(rn+n-e_1-\cdots-e_r)+b+1 \leq 0$. Hence the assertion is proved. □

Example 2.9 ([11])**.** Let $Y = \mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K$ be the Segre embedding in $\mathbb{P}^9_K$. Let X be an irreducible reduced divisor linearly equivalent to $p_1^*\mathcal{O}_{\mathbb{P}^1_K}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_K}(a+b) \otimes p_3^*\mathcal{O}_{\mathbb{P}^1_K}(a+2b)$, where $a \geq 1$ and $b \geq 2$. Then $k(X) = b$ and $\tilde{k}(X) > b$.

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Department of Mathematical Sciences, University of the Ryukyus, Nishihara-cho, Okinawa 903-0213 Japan
E-mail address: miyazaki@math.u-ryukyu.ac.jp