PROJECTIVE CURVES WITH NEXT TO SHARP BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY

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Abstract. This paper is devoted to the study of the next extremal case for a Castelnuovo-type bound $\operatorname{reg} C \leq \lceil(\deg C - 1)/\operatorname{codim} C\rceil + \max\{k(C), 1\}$ for the Castelnuovo-Mumford regularity for a nondegenerate projective curve C, where $k(C)$ is an invariant which measures the deficiency of the Hartshorne-Rao module of C. We show that a projective curve with next to the maximal regularity lies on either a Hirzebruch surface or a normal del Pezzo surface.

1. Introduction

Let k be an algebraically closed field. Let $\mathbb{P}_k^N = \operatorname{Proj} S$ be the projective N-space, where S is the polynomial ring of $N+1$ variables over k. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_k^N$ and an integer $m \in \mathbb{Z}$, $\mathcal{F}$ is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. For a projective scheme $X \subseteq \mathbb{P}_k^N$, X is said to be m-regular if the ideal sheaf $\mathcal{I}_X$ is m-regular. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_k^N$ is the least such integer m and is denoted by $\operatorname{reg} X$. It is well-known that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the pth syzygy module of the defining ideal $I(\subseteq S)$ of $X \subseteq \mathbb{P}_k^N$ occur in degree $\leq m+p$. In this sense, the Castelnuovo-Mumford regularity is one of the important invariants measuring a complexity of the defining ideal of a given projective scheme.

Throughout this paper, a curve is always assumed to be irreducible and reduced. For a rational number $m \in \mathbb{Q}$, we write $\lceil m \rceil$ for the minimal integer which is larger than or equal to m, and $\lfloor m \rfloor$ for the maximal integer which is smaller than or equal to m.

In this paper, we investigate a Castelnuovo-type bound for the Castelnuovo-Mumford regularity for projective curves. If a nondegenerate projective curve C is ACM, that is, the coordinate ring of C is Cohen-Macaulay, then there is a well-known inequality $\operatorname{reg} C \leq \lceil(\deg C - 1)/\operatorname{codim} C\rceil + 1$. The inequality follows from the fact that $\operatorname{reg} X \leq \lceil(\deg X - 1)/\operatorname{codim} X\rceil + 1$ for a generic hyperplane section X of C, which is an easy consequence of the Uniform Position Principle, see, e.g. [1, page 115] and [3, page 95], for characteristic zero. This also works for the general

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case, see, e.g., [14, (1.1)] from the property (2.1) of the h-vectors of X. The extremal case is described as a rational normal curve under the assumption $\deg C$ large enough, see [16]. In order to extend a result of Castelnuovo-type regularity bound for a (not necessarily ACM) curve, we introduce, as in [11, 12], an invariant $k(C)$ which measures how far the coordinate ring of C from the Cohen-Macaulay property. For a projective curve $C \subseteq \mathbb{P}^N_k$, a graded S-module $\mathrm{M}(C) = \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}^n_k}) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^1(\mathbb{P}^N_K, \mathcal{I}_C(\ell))$ is called the Hartshorne-Rao module. Then we define $k(C)$ as the minimal nonnegative integer v such that $\mathfrak{m}^v \mathrm{M}(C) = 0$. A curve C is ACM if and only if $k(C) = 0$. On the other hand, the coordinate ring of C is a Buchsbaum ring if and only if $k(C) = 1$. The extremal bound for the Buchsbaum curve, even for higher dimensional case, is also described in [17, 19]. For the general case, that is, C is a (not necessarily smooth) nondegenerate projective curve, we have an inequality $\operatorname{reg} C \leq \lceil (\deg C - 1)/\operatorname{codim} C \rceil + \max\{k(C), 1\}$, see (2.5). Furthermore, the following result (1.1) describes the extremal curve with the Castelnuovo-type maximal regularity from [3, (3.2)], or see [13, (1.2)].

Proposition 1.1. *Let $C \subseteq \mathbb{P}^N_k$ be a nondegenerate projective curve over an algebraically closed field k with* $\operatorname{char} k = 0$*. Assume that C is not ACM. If* $\deg C \geq (\operatorname{codim} C)^2 + 2 \operatorname{codim} C + 2$ *and* $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$*, then C lies on a rational normal surface scroll, that is, a Hirzebruch surface.*

The purpose of this paper is to study projective curves with next to sharp bounds of Castelnuovo-type on the Castelnuovo-Mumford regularity.

Theorem 1.2. *Let C be a nondegenerate projective curve over an algebraically closed field k with* $\operatorname{char} k = 0$*. Assume that C is not ACM, and* $\deg C \geq \max\{(\operatorname{codim} C)^2 + 4 \operatorname{codim} C + 2, 13\}$*. If*

$$\operatorname{reg} C = \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C) - 1,$$

then C lies either on a rational normal surface scroll or a normal del Pezzo surface.

Section 2 is devoted to the proof of (1.2). The theorem states that a curve with next to the maximal regularity of Castelnuovo-type corresponds to a divisor on either a rational normal surface scroll or a del Pezzo surface. Invariants of the divisor on a rational normal surface scroll concerning the inequality are calculated to describe the curve with maximal regularity in [13]. On the other hand, a classical del Pezzo surface is defined to be a smooth surface $V (\subseteq \mathbb{P}^N_k)$ with $\deg V = \operatorname{codim} V + 2$ such that $\omega_V \cong \mathcal{O}_V(-1)$ is either the blowups of general $d (\leq 6)$ points of $\mathbb{P}^2_k$ or the 2-uple embbeding of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^8_k$, see, e.g. [7, (4.7.1)]. A (not necessarily smooth) del Pezzo surface is classified by Fujita [4] and [5, (1.9.14)], see, e.g., [5, (1.6.3)] for the definition. In Section 3, we study some examples of divisors on a del Pezzo surface satisfying the equality in (1.2).

2. Proof of the main theorem

Let us introduce the terminology for the zero-dimensional scheme. Let $X \subseteq \mathbb{P}^N_k$ be a reduced zero-dimensional scheme such that X spans $\mathbb{P}^N_k$ as k-vector space. Then X is said to be in uniform position if $\mathrm{H}_Z(t) = \max\{\deg Z, \mathrm{H}_X(t)\}$ for all t, for any subscheme Z of X, where H_Z and H_X denote the Hilbert function of Z and X respectively. Let R be the coordinate ring of a zero-dimensional scheme $X \subseteq \mathbb{P}^N_k$. Let $\underline{h} = \underline{h}(X) = (h_0, \cdots, h_s)$ be the h-vector of $X \subseteq \mathbb{P}^N_k$, where $h_i = \dim_k[R]_i - \dim_k[R]_{i-1}$ and s is the largest integer such that $h_s \neq 0$. Note that $s = \operatorname{reg} X - 1$.

Remark 2.1. For a generic hyperplane section X of a projective curve, $h_1 + \cdots + h_i \geq ih_1$ for all $i = 1, \cdots, s-1$ by [2]. A generic hyperplane section of a nondegenerate projective curve is in uniform position if $\operatorname{char} k = 0$, see [1]. If X is in uniform position, then $h_i \geq h_1$ for $i = 1, \cdots, s-1$, see [10, Section 4].

In this section, from now on, let C be a nondegenerate projective curve of $\mathbb{P}^{N+1}_k$ and let H be a generic hyplerplane and $X = C \cap H \subseteq H \cong \mathbb{P}^N_k$. The following result (2.2) describes an extremal bound for the Castelnuovo-Mumford regularity of the generic hyperplane section of a projective curve $\operatorname{reg} X \leq \lceil (\deg X - 1)/N \rceil + 1$.

Lemma 2.2. (See [13, (2.6)]). *Let $X \subseteq \mathbb{P}^N_k$ be a generic hyperplane section of a nondegenerate projective curve. Assume that X is in uniform position and* $\deg X \geq N^2 + 2N + 2$. *If the equality* $\operatorname{reg} X = \lceil (\deg X - 1)/N \rceil + 1$ *holds, then X lies on a rational normal curve in $\mathbb{P}^N_k$.*

The extremal bound of the Castelnuovo-Mumford regularity for the generic hyperplane section of projective curve corresponds to a rational normal curve. The following lemma, which is obtained from Castelnuovo theory [8, Section 3], yields that the next extremal one corresponds to an elliptic normal curve.

Lemma 2.3. *Let $X \subseteq \mathbb{P}^N_k$ be a generic hyperplane section of a nondegenerate projective curve. Assume that X is in uniform position and* $\deg X \geq N^2 + 4N + 2$. *If the equality* $\operatorname{reg} X = \lceil (\deg X - 1)/N \rceil$ *holds, then X lies on either a rational normal curve or an elliptic normal curve in $\mathbb{P}^N_k$.*

Proof. Let $(h_0, \cdots, h_s)$ be the h-vector of the one-dimensional graded ring R. Note that $h_0 = 1$, $h_1 = N$ and $\deg X = h_0 + \cdots + h_s$. Suppose that X does not lie either on a rational normal curve or on an elliptic normal curve. Let us show that $h_2 \geq h_1 + 2$, with keeping in mind the fact $h_2 \geq h_1$ by (2.1). First, let us assume that $h_2 = h_1$, that is, $\dim_k[R]_2 = 2N + 1$. Since X is in uniform position, then X is contained in a rational normal curve by [8, (3.9)], which contradicts the hypothesis. Next, let us assume that $h_2 = h_1 + 1$, that is, $\dim_k[R]_2 = 2N + 2$. Since X is in uniform position and $\deg X \geq N^2 + 4N + 2 \geq 2N + 5$, X lies on a rational normal surface scroll by [8, (3.19)]. This implies that X is contained in an elliptic normal curve by [8, (3.20)], which contradicts the hypothesis. Hence we have $h_2 \geq h_1 + 2$. Since X is in uniform position, X is of decreasing type, see, e.g., [6]. Hence we have that $h_i \geq h_1 + 2$ for all $2 \leq i \leq s-3$, $h_{s-2} \geq h_1 + 1$ and $h_{s-1} \geq h_1$. Thus we have

$$\frac{\deg X - 1}{N} = \frac{h_1 + \cdots + h_s}{h_1}$$
$$\geq 1 + \overbrace{\frac{N+2}{N} + \cdots + \frac{N+2}{N}}^{s-4} + \frac{N+1}{N} + 1 + \frac{1}{N} \quad = s - 1 + \frac{2s-6}{N}.$$

Since $s + 1 \geq (\deg X - 1)/N$, we see that $N \leq s - 3$. Hence we have $\deg X - 1 \leq N(s+1) \leq N^2 + 4N$, which contradicts the hypothesis. □

Remark 2.4. In the statement of (1.1), we may take an assumption that $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ for a generic hyperplane section X of C in place of the equality $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$.

Proposition 2.5 ([18])**.** *Let $C \subseteq \mathbb{P}_k^{N+1}$ be a nondegenerate projective curve over an algebraically closed field. Assume that C is not ACM. Then*

$$\operatorname{reg} C \leq \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C).$$

Proof. The assertion is a consequence of [18]. However, in order to use the process in the proof of (1.2), we will give a short proof. Let $X = C \cap H$ be a generic hyperplane section. Let $m = \operatorname{reg} X$. Let $n = k(C)$. From the exact sequence

$$\begin{array}{cccccc} & \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) & \xrightarrow{\cdot h} & \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}) & \to & \mathrm{H}^1_*(\mathcal{I}_{X/H}) \\ \to & \mathrm{H}^2_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) & \xrightarrow{\cdot h} & \mathrm{H}^2_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}), & & \end{array}$$

where h is a defining equation of H, we have $\mathrm{h}^2(\mathcal{I}_{\mathcal{C}}(m-2)) \leq \mathrm{h}^2(\mathcal{I}_{\mathcal{C}}(m-1)) \leq \cdots \leq 0$ and $\mathrm{H}^1(\mathcal{I}_C(m+n-2)) = h \cdot \mathrm{H}^1(\mathcal{I}_C(m+n-3)) = \cdots = h^n \cdot \mathrm{H}^1(\mathcal{I}_C(m-2)) = 0$. Hence we obtain $\operatorname{reg} C \leq \operatorname{reg} X + n - 1 \leq \lceil (\deg X - 1)/N \rceil + k(C) = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$. □

Proof of Theorem 1.2. Let C be a nondegenerate projective curve in $\mathbb{P}_k^{N+1} = \operatorname{Proj} S$, where S be the polynomial ring and $\mathfrak{m}$ is the irrelevant ideal. Let $X = C \cap H$ be a generic hyperplane section. From the last line of the proof of (2.5), the equality $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$ gives either $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ or $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil$. By (2.2) and (2.3), X lies on either (i) a rational normal curve, or (ii) an elliptic normal curve. For the case (i), C is contained in a rational normal surface scroll from (1.1) and (2.4). Thus we are done in this case. Let us consider the case (ii). We may assume that X is contained in an elliptic normal curve Z in $H (\cong \mathbb{P}_k^N)$. Let $c = \operatorname{codim} C$ and $d = \deg C$. Then $\deg X = d$, $\operatorname{codim} X = c + 1$ and $\deg Z = \operatorname{codim} Z + 2 = c + 2$. For $c = 1$, Z is a plane smooth cubic curve. For $c \geq 2$, Z is generated by quadric equations.

First, we will show that $\Gamma(\mathcal{I}_{Z/H}(2)) \cong \Gamma(\mathcal{I}_{X/H}(2))$ if $c \geq 2$ and $\Gamma(\mathcal{I}_{Z/H}(3)) \cong \Gamma(\mathcal{I}_{X/H}(3))$ if $c = 1$. Indeed, for $c \geq 2$, if there exists a hyperquadric Q such that $X \subseteq Q$ and $Z \nsubseteq Q$, then $X \subseteq Z \cap Q$ and $d \leq 2(c+2)$ by Bezout theorem, which contradicts the assumption $d \geq c^2 + 4c + 2$. For the case $c = 1$, we obtain an isomorphism $\Gamma(\mathcal{I}_{Z/H}(3)) \cong \Gamma(\mathcal{I}_{X/H}(3))$. In fact, if not, then an inequality $d \leq 3(c+3)$ similarly obtained from Bezout theorem contradicts the assumption $d \geq 13$.

Next, we will show that $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective if $c \geq 2$, and $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(3)) \to \Gamma(\mathcal{I}_{X/H}(3))$ is surjective if $c = 1$. Indeed, let $\varphi : \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) \xrightarrow{\cdot h} \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})$, where $h(\in [S]_1)$ is a linear form defining the hyperplane H. From the exact sequence

$$\begin{array}{rcl} & \Gamma_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}) \to & \Gamma_*(\mathcal{I}_{X/H}) \\ \to \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) & \xrightarrow{\varphi} \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}) \to & \mathrm{H}^1_*(\mathcal{I}_{X/H}), \end{array}$$

we need to prove that $[\operatorname{Ker}\varphi]_2 = 0$ if $c \geq 2$, and $[\operatorname{Ker}\varphi]_3 = 0$ if $c = 1$. Then we see that $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective if $c \geq 2$, and $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(3)) \to \Gamma(\mathcal{I}_{X/H}(3))$ is surjective if $c = 1$. By the Socle Lemma[9, (3.11)], for a generic linear form $h \in [S]_1$ we have $a_-(\operatorname{Ker}\varphi) > a_-(\operatorname{Coker}\varphi)$, where $\operatorname{Soc}(N)$ is the set of elements of N annihilated by the maximal ideal $\mathfrak{m}$ and $a_-(N) = \min\{\ell | [N]_\ell \neq 0\}$ for a graded S-module N. Hence we have $a_-(\operatorname{Ker}\varphi) > a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H})))$.

Now let us evaluate $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H})))$. Since Z is ACM, we have the short exact sequence

$$0 \to \mathrm{H}^1_*(\mathcal{I}_{X/H}) \to \mathrm{H}^1_*(\mathcal{I}_{X/Z}) \to \mathrm{H}^2_*(\mathcal{I}_{Z/H}) \to 0$$

from the short exact sequence $0 \to \mathcal{I}_{Z/H} \to \mathcal{I}_{X/H} \to \mathcal{I}_{X/Z} \to 0$. Note that $\mathrm{H}^2_*(\mathcal{I}_{Z/H}) \cong \mathrm{H}^1_*(\mathcal{O}_Z) \cong k$. Now we will investigate the structure of $\mathrm{H}^1_*(\mathcal{I}_{X/Z})$. By Serre duality, $\mathrm{H}^1_*(\mathcal{I}_{X/Z})$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_Z(X))$. Hence $\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/Z}))$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_Z(X))/\mathfrak{m}\Gamma_*(\mathcal{O}_Z(X))$. Let $\mathcal{F} = \mathcal{O}_Z(X)$. Since Z is a smooth elliptic curve, we see that $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_Z(m-1)) = 0$ if $-d-(m-1)(c+2) < 0$. In other words, $\mathcal{F}$ is m-regular for $m \geq (c-d+3)/(c+2)$. Let $m = \lceil (c-d+3)/(c+2) \rceil$. Then we see that

$$\Gamma(\mathcal{F} \otimes \mathcal{O}_Z(\ell)) \otimes \Gamma(\mathcal{O}_Z(1)) \to \Gamma(\mathcal{F}(\ell+1))$$

is surjective for $\ell \geq m$ by [15]. Hence we obtain $a_-(\operatorname{Soc}(\mathrm{H}^1_*\mathcal{I}_{X/Z})) \geq -m$. Therefore, if $d \geq 3c+7$, then $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H}))) \geq 2$, and if $d \geq 4c+9$, $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H}))) \geq 3$. Since $d \geq \max\{c^2+4c+2, 13\}$, we obtain $[\operatorname{Ker}\varphi]_2 = 0$ if $c \geq 2$ and $[\operatorname{Ker}\varphi]_3 = 0$ if $c = 1$.

For the case $c \geq 2$, we have a surjective map $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2)) \cong \Gamma(\mathcal{I}_{Z/H}(2))$. Note that Z is the intersection of the hyperquadrics containing X. Let Y' be the intersection of the hyperquadrics containing C. Since $Y' \cap H = Z$, there is an irreducible component Y of Y' such that $Y \cap H = Z$. For the case $c = 1$, we are similarly done as

$c \geq 2$. Thus there exists a surface Y containing C such that $Y \cap H = Z$ and $\deg Y = \operatorname{codim} Y + 2$. Since a hyperplane section is an elliptic normal curve, Y is a normal surface. By [5, (1.6.5)], Y must be a normal del Pezzo surface. □

Remark 2.6. Although I do not have counterexamples for the main theorem without the degree condition, the assumption $\deg C \gg 0$ seems to be indispensable. In fact, a non-hyperelliptic curve of genus $g \geq 5$ with the canonical embedding satisfies the extremal bound for ACM case, but is not in a surface of minimal degree, see [19, page 160]. Moreover, there is a counterexample for (2.2) without degree condition, see [13, (2.6)].

3. Examples

Before studying a curve on a del Pezzo surface, we describe a curve on a rational normal surface scroll with next to the extremal regularity. The proof is similar as [13, (1.5)].

Example 3.1. Let $\pi : V = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_k$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-e)$ for some $e \geq 0$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1_k}(-e)$ and F be a fibre corresponding to $\pi^*\mathcal{O}_{\mathbb{P}^1_k}(1)$. We have an embedding of V in $\mathbb{P}^N_K$ by a very ample sheaf corresponding to a divisor $H = Z + n \cdot F$ $(n > e)$, where $N = 2n - e + 1$. Let C be a divisor C on V linearly equivalent to $a \cdot Z + b \cdot F$ such that $a \geq 1$ and $(a+2)n - e + 2 \leq b \leq (a+2)n - e + 1 + (2n - e)$. Then we see that $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C) - 1$.

In particular, in case, $e = 0$, that is, $V (\cong \mathbb{P}^1_k \times \mathbb{P}^1_k)$ is a smooth quadric surface in $\mathbb{P}^3_k$. Let C be a divisor on V of type (a, b). The curve C satisfies the next extremal bound if and only if $4 \leq |b - a| \leq 5$. So, there exists curves with next extremal bound even if the genus $(a-1)(b-1)$ is higher.

Now, we will study projective curves on some smooth del Pezzo surfaces with next to the extremal regularity.

Example 3.2. Let $V = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Let π_1 and π_2 be the first and second projection respectively. We write $\mathcal{O}_V(a, b)$ for $\pi_1^*\mathcal{O}_V(a) \otimes \pi_2^*\mathcal{O}_V(b)$. Let Z_1 and Z_2 be divisors corresponding to $\mathcal{O}_V(1, 0)$ and $\mathcal{O}_V(0, 1)$ respectively. We have a 2-uple embedding of V by $H = 2Z_1 + 2Z_2$. Then V is a del Pezzo surface of degree 8 in $\mathbb{P}^8_k$. Let C be a divisor on V linearly equivalent to $a \cdot Z_1 + b \cdot Z_2$. We may assume $a \leq b$. By calculating the cohomologies $\mathrm{H}^i(\mathcal{I}_{C/V}(\ell H)) \cong \mathrm{H}^i(\mathcal{O}_V(-a + 2\ell, -b + 2\ell))$, $i = 1, 2$, by Künneth formula, we see that $[\mathrm{H}^1]_\ell \neq 0$ if and only if $a/2 \leq \ell \leq (b-2)/2$, and $[\mathrm{H}^2]_\ell \neq 0$ if and only if $\ell \leq (a-2)/2$. Assume that C is not ACM. Then we have $b \geq a+2$. In this case, we have $k(C) = \lfloor b/2 \rfloor - \lceil a/2 \rceil$, and $\operatorname{reg} C = \lfloor b/2 \rfloor + 1$. Also, we have $\deg C = 2a + 2b$. Thus there exists a curve C on V satisfying $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ by choosing a and b such that $\lceil (a+4)/2 \rceil = \lceil (2a + 2b - 1)/7 \rceil$, while there are no such curves for $k(C)$ large enough.

Example 3.3. Let $\pi : V = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_k$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1_k}(-1)$ and F be a fibre corresponding to π. We have an embedding of V in $\mathbb{P}^8_k$ by a very ample sheaf corresponding to a divisor $H = 2 \cdot Z + 3 \cdot F$. Then V is a del Pezzo surface of degree 8 in $\mathbb{P}^8_k$. Let C be a divisor on V linearly equivalent to $a \cdot Z + b \cdot F$. From [13, (2.12)], $\mathrm{H}^1(V, \mathcal{O}_V(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if either $\alpha \geq 0$ and $\beta \leq \alpha - 2$, or $\alpha \leq -2$ and $\beta \geq \alpha + 1$. Thus $\mathrm{H}^1(\mathcal{I}_{C/V}(\ell H)) \neq 0$ if and only if either $a/2 \leq \ell \leq -a + b - 2$ or $-a + b + 1 \leq \ell \leq (a-2)/2$. From [13, (2.14)], $\mathrm{H}^2(V, \mathcal{O}_V(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\alpha \leq -2$ and $\beta \leq -3$. Thus $\mathrm{H}^2(\mathcal{I}_{C/V}(\ell H)) \neq 0$ if and only if $-a + 2\ell \leq -2$ or $-b + 3\ell \leq -3$. Hence we have $k(C) = b - \lceil 3a/2 \rceil - 1$ for $b \geq 3a/2 + 2$, and $k(C) = \lfloor 3a/2 \rfloor - b + 3$ for $b \leq 3a/2 + 2$. On the other hand, we have $\operatorname{reg} C = b - a$ for $b \geq 3a/2 + 2$, $\operatorname{reg} C = \lfloor a/2 \rfloor + 2$ for $3a/2 \leq b \leq 3a/2 + 2$, and $\operatorname{reg} C = \lfloor b/3 \rfloor + 2$ for $b \leq 3a/2$. Also, we have $\deg C = a + 2b$. For $b \leq 3a/2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lfloor 4b/3 \rfloor = \lceil (a + 2b + 6)/7 \rceil + \lfloor 3a/2 \rfloor$ which does not happen for this case. For $3a/2 < b < 3a/2 + 2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lceil (8a - 5b - 1)/7 \rceil = 0$, which does not happen if $\deg C \geq 79$. For $b \geq 3a/2 + 2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lceil a/2 \rceil = \lceil (a + 2b - 15)/7 \rceil$. In this case, there exists a curve C on V satisfying $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ by choosing a and b with $b \geq 3a/2 + 2$ such that $\lceil a/2 \rceil = \lceil (a + 2b - 15)/7 \rceil$, while there are no such curves with $k(C)$ large enough.

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