Cohomological Property of Vector Bundles on Biprojective Spaces

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Abstract

This paper investigates the cohomological property of vector bundles on biprojective space. We will give a criterion for a vector bundle to be isomorphic to the tensor product of pullbacks of exterior products of differential sheaves.

1 Introduction

The purpose of this paper is to study the cohomological property of vector bundle towards Horrocks-type criteria and to characterize the tensor product of pullbacks of exterior products of differential sheaves on biprojective space. Horrocks Theorem says that an ACM vector bundle on the projective space is isomorphic to a direct sum of line bundles. There are some attempts to generalize to the biprojective space, that is, some splitting criteria for a vector bundle on $\mathbb{P}^m \times \mathbb{P}^n$ to be isomorphic to a direct sum of the form $\mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n}(\ell_1, \ell_2)$ in [2, 4, 6]. In particular in [4] it is used a Beilinson type spectral sequence and m-blocks collection while in [2, 6] it is used a notion of Castelnuovo-Mumford regularity and Koszul complexes (for a similar approach on Grassmannians see [1]). In this paper we will give a cohomological criterion for a vector bundle on $\mathbb{P}^m \times \mathbb{P}^n$ to have a direct summand of the form $\Omega^p_{\mathbb{P}^m} \boxtimes \Omega^q_{\mathbb{P}^n}$ using the second approach.

Let us describe our perspective on the condition for a vector bundle to have a specific direct summand. We will begin a proof of the Horrocks theorem through the Castelnuovo-Mumford regularity according to [2]. Let E be an ACM vector bundle on $\mathbb{P}^n$. Assume that E is m-regular but not $(m-1)$-regular, see [8] for the definition and basic properties for m-regular. Then we have a surjective map $\varphi : \mathcal{O}^{\oplus}_{\mathbb{P}^n} \to E(m)$. Since E is ACM, we have $\mathrm{H}^n(E(m-1-n)) \neq 0$, and $\mathrm{H}^0(E^\vee(-m)) \neq 0$ by Serre duality. Thus we have a nonzero map $\psi : E(m) \to \mathcal{O}_{\mathbb{P}^n}$. Since $\psi \circ \varphi$ is nonzero, it splits. Hence $\mathcal{O}_{\mathbb{P}^n}$ is a direct summand of $E(m)$.

Now we will proceed the next step on studying a Buchsbaum vector bundle on $\mathbb{P}^n$, that is, $\mathfrak{m}\mathrm{H}^i_*(E|_L) = 0$ for any r-plane L of $\mathbb{P}^n$, $1 \le i < r \le n$, where $\mathfrak{m} = \oplus_{\ell\ge 1}\Gamma(\mathcal{O}_{\mathbb{P}^n}(\ell))$. Instead of using the regularity we will make use of the Koszul complex. Before giving a sufficient condition

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for a vector bundle to be a direct sum of vector bundles of the form $\Omega^i_{\mathbb{P}^n}(\ell)$, we will describe important facts concerning the structure of Buchsbaum vector bundle.

Proposition 1.1 ([3, 5])**.** *Let E be a Buchsbaum vector bundle on $\mathbb{P}^n$. Then E is isomorphic to a direct sum of vector bundles of the form $\Omega^i_{\mathbb{P}^n}(\ell)$.*

Proposition 1.2 ([9], (I.3.10))**.** *Let E be a vector bundle on $\mathbb{P}^n$. Let us define $\mathfrak{S} = \{(i,\ell) | 1 \le i \le n-1, \ell \in \mathbb{Z}, \mathrm{H}^i(E(\ell)) \neq 0\}$. Suppose that $\mathfrak{S}$ satisfies the following condition: "For $(i,\ell),(j,m) \in \mathfrak{S}$, if $i \ge j$, then $i+\ell+1 \neq j+m$". Then E is Buchsbaum.*

From these facts we have observed the relation between the Buchsbaum property and the differential sheaves in (1.1) and a cohomological characterization of Buchsbaum modules in (1.2). Then we will give a straightforward proof of a more or less known result which illustrates the relation between the vanishings of the intermediate cohomologies and the exterior products of differential sheaves. The following is a starting point of our main result, Theorem 2.1.

Proposition 1.3. *Let E be a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^p(E) \neq 0$, where $1 \le p \le n-1$. If a vector bundle E has the following condition:*

(a) $\mathrm{H}^i(E(p-i+1)) = 0$ *for* $1 \le i \le p$.

(b) $\mathrm{H}^i(E(p-i-1)) = 0$ *for* $p \le i \le n-1$,

then E contains $\Omega^p_{\mathbb{P}^n}$ as a direct summand.

Proof. By an exact sequence arising from the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(1) \to \cdots \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(p) \to \Omega^{p\vee}_{\mathbb{P}^n} \to 0,$$

we have a surjective map $\varphi : \mathrm{H}^0(E \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \to \mathrm{H}^p(E)$ from the assumption $\mathrm{H}^1(E(p)) = \cdots = \mathrm{H}^p(E(1)) = 0$. By an exact sequence arising from the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(-n) \to \cdots \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(-p-1) \to \Omega^p_{\mathbb{P}^n} \to 0,$$

we have a surjective map $\psi : \mathrm{H}^0(E^\vee \otimes \Omega^p_{\mathbb{P}^n}) \to \mathrm{H}^{n-p}(E^\vee(-n-1))$ from the assumption $\mathrm{H}^p(E(-1)) = \cdots = \mathrm{H}^{n-1}(E(p-n)) = 0$, that is, $\mathrm{H}^1(E^\vee(-p-1)) = \cdots = \mathrm{H}^{n-p}(E^\vee(-n)) = 0$.

As in [2] we have a nonzero element $f \in \mathrm{H}^0(E \otimes \Omega^{p\vee}_{\mathbb{P}^n})$ such that $\varphi(f) = s (\neq 0) \in \mathrm{H}^p(E)$. Let us take an element $s^* \in \mathrm{H}^{n-p}(E^\vee(-n-1))$ corresponding to $s \in \mathrm{H}^m(E)$, there is a nonzero element $g \in \mathrm{H}^0(E^\vee \otimes \Omega^p_{\mathbb{P}^n})$ such that $\psi(g) = s^* (\neq 0) \in \mathrm{H}^{n-p}(E^\vee(-n-1))$. Then f and g are regarded as elements of $\mathrm{Hom}(\Omega^p_{\mathbb{P}^n}, E)$ and $\mathrm{Hom}(E, \Omega^p_{\mathbb{P}^n})$ respectively. From a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^0(E \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \otimes \mathrm{H}^0(E^\vee \otimes \Omega^p_{\mathbb{P}^n}) & \to & \mathrm{H}^0(\Omega^{p\vee}_{\mathbb{P}^n} \otimes \Omega^p_{\mathbb{P}^n}) \cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}) \\ \downarrow & & \downarrow \\ \mathrm{H}^p(E) \otimes \mathrm{H}^{n-p}(E^\vee(-n-1)) & \to & \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)), \end{array}$$

a natural map $\mathrm{H}^0(E \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \otimes \mathrm{H}^0(E^\vee \otimes \Omega^p_{\mathbb{P}^n}) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n})$ yields that $g \circ f$ is an isomorphism, which implies $\Omega^p_{\mathbb{P}^n}$ is a direct summand of E. □

2 Cohomological Criterion of Vector Bundles on Biprojective Space

What condition is required for a vector bundle E on $\mathbb{P}^m \times \mathbb{P}^n$ to have a direct summand of the form $\Omega^p_{\mathbb{P}^m} \boxtimes \Omega^q_{\mathbb{P}^n}$? Although the exterior products of differential sheaves are the indecomposable

Buchsbaum vector bundles on $\mathbb{P}^n$, The Buchsbaum property of differential sheaves are more complicated on $\mathbb{P}^m \times \mathbb{P}^n$, see [7]. This section is devoted to an answer of cohomological criteria of differential sheaves from the viewpoint of (1.3). Compared with an important result of [4, (4.11)], our theorem obtained from an elementary way concludes an isomorphism to just one bundle directly.

Theorem 2.1. *Let E be a vector bundle on $\mathbb{P}^m \times \mathbb{P}^n$ with $\mathrm{H}^{p+q}(E) \neq 0$, where $1 \leq p \leq m-1$ and $1 \leq q \leq n-1$. If a vector bundle E has the following condition:*

(a) $\mathrm{H}^i(E(a,b)) = 0$ *for* $1 \leq i \leq p+q$, $0 \leq a \leq p$, $0 \leq b \leq q$ *with* $i+a+b = p+q+1$.

(b) $\mathrm{H}^i(E(a,b)) = 0$ *for* $p+q \leq i \leq m+n-1$, $p-m \leq a \leq 0$, $q-n \leq b \leq 0$ *with* $i+a+b = p+q-1$,

then E contains $\Omega^p_{\mathbb{P}^m} \boxtimes \Omega^q_{\mathbb{P}^n}$ as a direct summand.

Proof. Let us consider the exact sequences arising from the Koszul complexes

$$0 \to \Omega^p_{\mathbb{P}^m} \to \mathcal{O}^{f_p}_{\mathbb{P}^m}(-p) \to \mathcal{O}^{f_{p-1}}_{\mathbb{P}^m}(-p+1) \to \cdots \to \mathcal{O}^{f_1}_{\mathbb{P}^m}(-1) \to \mathcal{O}_{\mathbb{P}^m} \to 0 \tag{1}$$

and

$$0 \to \Omega^q_{\mathbb{P}^n} \to \mathcal{O}^{e_q}_{\mathbb{P}^n}(-q) \to \mathcal{O}^{e_{q-1}}_{\mathbb{P}^n}(-q+1) \to \cdots \to \mathcal{O}^{e_1}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0, \tag{2}$$

where $f_i = \begin{pmatrix} m \\ i \end{pmatrix}$ and $e_j = \begin{pmatrix} n \\ j \end{pmatrix}$. By gluing the pull back by $p_2 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ of the dual of (2) tensored by E and the pull back by $p_1 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ of the dual of (1) tensored by $E \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n}$ we obtain an exact sequence

$$0 \to E \to E(0,1)^{e_1} \to E(0,2)^{e_2} \to \cdots \to E(0,q)^{e_q} \to E(1,0)^{f_1} \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n}$$

$$\to E(2,0)^{f_2} \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n} \to \cdots \to E(p,0)^{f_p} \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n} \to E \otimes p_1^*\Omega^{p\vee}_{\mathbb{P}^m} \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n} \to 0.$$

Notice that

$$\Omega^{q\vee}_{\mathbb{P}^n}. \cong \Omega^{n-q}_{\mathbb{P}^n}(n+1)$$

In order to have a surjective map

$$\varphi : \mathrm{H}^0(E \otimes p_1^*\Omega^{p\vee}_{\mathbb{P}^m} \otimes p_2^*\Omega^{q\vee}_{\mathbb{P}^n}) \to \mathrm{H}^{p+q}(E),$$

we will show

(c.1) $\mathrm{H}^1(E(p,n+1) \otimes p_2^*\Omega^{n-q}_{\mathbb{P}^n}) = \cdots = \mathrm{H}^p(E(1,n+1) \otimes p_2^*\Omega^{n-q}_{\mathbb{P}^n}) = 0.$

(c.2) $\mathrm{H}^{p+1}(E(0,q)) = \cdots = \mathrm{H}^{p+q}(E(0,1)) = 0.$

The asserion (c.2) follows from the assumption (a). Since $\mathrm{H}^i(E(p-i+1,q)) = \mathrm{H}^{i+1}(E(p-i+1,q-1)) = \cdots = \mathrm{H}^{i+q}(E(p-i+1,0)) = 0$, $i = 1, \cdots, p$, we see that $\mathrm{H}^i(E(p-i+1,n+1) \otimes p_2^*\Omega^{n-q}_{\mathbb{P}^n}) = 0$ from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^{e_n}_{\mathbb{P}^n}(1) \to \cdots \to \mathcal{O}^{e_{n-q+1}}_{\mathbb{P}^n}(q) \to \Omega^{n-q}_{\mathbb{P}^n}(n+1) \to 0$$

by pulling back to p_2 and tensored by $E(p-i+1,0)$. Thus we obtain (c.1).

Next, let us consider the exact sequences arising from the Koszul complexes

$$0 \to \mathcal{O}_{\mathbb{P}^m}(-m-1) \to \mathcal{O}^{f_m}_{\mathbb{P}^m}(-m) \to \cdots \to \mathcal{O}^{f_{p+1}}_{\mathbb{P}^m}(-p-1) \to \Omega^p_{\mathbb{P}^m} \to 0 \tag{3}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}^{e_n}(-n) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{e_{q+1}}(-q-1) \to \Omega_{\mathbb{P}^n}^q \to 0 \tag{4}$$

By gluing the pull back by p_2 of (4) tensored by $E^\vee(-m-1,0)$ and the pull back by p_1 of (3) tensored by $E^\vee \otimes p_2^*\Omega_{\mathbb{P}^n}^q$ we obtain an exact sequence

$$\begin{aligned} &0 \to E^\vee(-m-1,-n-1) \to E^\vee(-m-1,-n)^{e_m} \to \cdots \\ &\cdots \to E^\vee(-m-1,-q-1)^{e_{p+1}} \to E^\vee(-m,0)^{f_m} \otimes p_2^*\Omega_{\mathbb{P}^n}^q \to \cdots \\ &\cdots \to E^\vee(-p-1,0)^{f_{p+1}} \otimes p_2^*\Omega_{\mathbb{P}^n}^q \to E^\vee \otimes p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q \to 0. \end{aligned}$$

In order to have a surjective map

$$\psi : \mathrm{H}^0(E^\vee \otimes p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q) \to \mathrm{H}^{m+n-p-q}(E^\vee(-m-1,-n-1))$$

we will show that $\mathrm{H}^1(E^\vee(-p-1,0) \otimes p_2^*\Omega_{\mathbb{P}^n}^q) = \cdots = \mathrm{H}^{m-p}(E^\vee(-m,0) \otimes p_2^*\Omega_{\mathbb{P}^n}^q) = 0$ and $\mathrm{H}^{m-p+1}(E^\vee(-m-1,-q-1)) = \cdots = \mathrm{H}^{m+n-p-q}(E^\vee(-m-1,-n) = 0$. By Serre duality, we have only to show

(d.1) $\mathrm{H}^{m+n-1}(E(p-m,0) \otimes p_2^*\Omega_{\mathbb{P}^n}^{n-q}) = \cdots = \mathrm{H}^{n+p}(E(-1,0) \otimes p_2^*\Omega_{\mathbb{P}^n}^{n-q}) = 0.$

(d.2) $\mathrm{H}^{n+p-1}(E(0,-n+q)) = \cdots = \mathrm{H}^{p+q}(E(0,-1)) = 0.$

The assertion (d.2) follows from the assumption (b). Since $\mathrm{H}^i(E(n+p-i-1,-n+q)) = \cdots = \mathrm{H}^{i-n+q}(E(n+p-i-1,0)) = 0$, $i = n+p, \cdots, m+n-1$, from the assumption (b), we see that $\mathrm{H}^i(E(n+p-i-1,0) \otimes p_2^*\Omega_{\mathbb{P}^n}^{n-q}) = 0$ from the exact sequence

$$0 \to \Omega_{\mathbb{P}^n}^{n-q} \to \mathcal{O}_{\mathbb{P}^n}^{e_{n-q}}(-n+q) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{e_1}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

by pulling back to p_2 and tensored by $E(n+p-i-1,0)$. Thus we obtain (d.1).

As in (1.3), for a nonzero element $s \in \mathrm{H}^{p+q}(E)$ and the corresponding element $s^* \in \mathrm{H}^{m+n-p-q}(E^\vee(-m-1,-n-1))$ by Serre duality, there are $f \in \mathrm{H}^0(E \otimes p_1^*\Omega_{\mathbb{P}^m}^{p\vee} \otimes p_2^*\Omega_{\mathbb{P}^n}^{q\vee})$ and $g \in \mathrm{H}^0(E^\vee \otimes p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q)$ with $\varphi(f) = s$ and $\psi(g) = s^*$. Then $g \circ f$ is an isomorphism by regarding as $f \in \mathrm{Hom}(p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q, E)$ and $g \in \mathrm{Hom}(E, p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q)$, which gives an inclusion from $p_1^*\Omega_{\mathbb{P}^m}^p \otimes p_2^*\Omega_{\mathbb{P}^n}^q$ to E as a direct summand. □

Example 2.2. Let us give an application of (2.1) to a vector bundle of $\mathbb{P}^2 \times \mathbb{P}^2$. Let E be an indecomposable vector bundle on $\mathbb{P}^2 \times \mathbb{P}^2$. Then the following conditions are equivalent:

(a) $E \cong \Omega_{\mathbb{P}^2} \boxtimes \Omega_{\mathbb{P}^2}$.

(b) $\mathrm{H}^2(E) \neq 0$ and $\mathrm{H}^1(E(1,1)) = \mathrm{H}^2(E(0,1)) = \mathrm{H}^2(E(1,0)) = \mathrm{H}^2(E(-1,0)) = \mathrm{H}^2(E(0,-1)) = \mathrm{H}^3(E(-1,-1)) = 0$.

References

[1] E. Arrondo and F. Malaspina, Cohomological characterization of vector bundles on Grassmannians of lines, J. Algebra 323 (2010), no. 4, 1098–1106.

[2] E. Ballico and F. Malaspina, Regularity and cohomological splitting conditions for vector bundles on multiprojective spaces, J. Algebra 345 (2011), 137 – 149.

[3] M. C. Chang, Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$, J. Differential Geom. 31 (1990), 323–341.

[4] L. Costa and R. M. Miró-Roig, Cohomological characterization of vector bundles on multiprojective spaces, J. Algebra 294 (2005), 73–96, with a corrigendum in J. Algebra 319 (2008), 1336–1338.

[5] S. Goto, Maximal Buchsbaum modules over regular local rings and a structure theorem for generalized Cohen-Macaulay modules, ASPM 11(1987), 39–64.

[6] C. Miyazaki, A cohomological criterion for splitting of vector bundles on multiprojective space, Proc. Amer. Math. Soc. 143 (2015), 1435–1440.

[7] C. Miyazaki, Buchsbaum criterion of Segre products of vector bundles on multiprojective space, J. Algebra 467 (2016), 47–57.

[8] D. Mumford, Lectures on curves on an algebraic surface, Annals of Math. Studies 59 (1966), Princeton UP.

[9] J. Stückrad and W. Vogel, Buchsbaum rings and applications, Springer, 1986.