Bijective proofs of the identities on the values of inner products of the Macdonald polynomials

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Introduction

The Hall-Littlewood polynomials $P_\lambda(t)$ are a family of symmetric polynomials indexed by partitions [2]. They are a generalization of the Schur polynomials having a parameter t. The Macdonald polynomials $P_\lambda(q,t)$ are a yet more generalization having two parameters q and t [3].

There are inner products defined on the space of symmetric polynomials with which the power sum symmetric polynomials form an orthogonal basis. These kind of inner products are introduced originally by Redfield [6] and Hall [1]. One obtains some identities of parameters by calculating the inner products of some symmetric polynomials.

In this article, we give alternative proofs of the following well-known identities:

$$\prod_{i=1}^{n} \frac{1-qt^{i-1}}{1-t^i} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},$$

$$\prod_{i=1}^{n} \frac{t^{i-1}-q}{1-t^i} = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

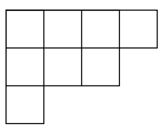
The identities are immediately obtained by using (2.14') and Example 5 in the Chapter I, Section 2 of [4]. We prove them by calculating the inner products of Hall-Littlewood polynomials and Macdonald polynomials, and by constructing certain bijections through transforming the Young diagrams of partitions.

This article consists of two sections. In Section 1, we introduce some concepts, and prove the identities in the case $q=0$, which have only one parameter t. The identities in this case are obtained by calculating the inner product of some symmetric polynomials including the Hall-Littlewood polynomials. In Section 2, we prove the identities in the general case, which have two parameters q, t. The identities are obtained from the inner product of symmetric polynomials including the Macdonald polynomials. In each section, we provide a bijective proof of the identity. In this article, $\mathbb{N}$ denotes the set of all non-negative integers.

1 The case of one parameter

A *partition* is a weakly decreasing finite series of positive integers. Let $\mathscr{P}$ denote the set of all partitions. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathscr{P}$, $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ is called the *weight* of λ and $l(\lambda) = l$ is the *length* of λ. Write $\mathscr{P}_n = \{\lambda \in \mathscr{P} \mid l(\lambda) \leq n\}$ and $\mathscr{P}'_n = \{\lambda \in \mathscr{P} \mid \lambda_1 \leq n\}$. If $|\lambda| = n$, λ is called a *partition of* n. We write $\lambda \vdash n$ if λ is a partition of n. Write $\mathscr{P}(n) = \{\lambda \in \mathscr{P} \mid \lambda \vdash n\}$.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathscr{P}$, the *Young diagram* of λ is the diagram consisting of l rows of left-aligned cells, the ith row from the top has λ_i cells. For example, $(4,3,1)$ is a partition of 8 and its Young diagram is as follows.



For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathscr{P}$ and $i \geq 1$, $m_i(\lambda) = \#\{j \mid \lambda_j = i\}$ is called the *multiplicity* of i in λ. A partition $\lambda \in \mathscr{P}$ is also written as $\lambda = \left(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots\right)$ by using the multiplicities. We define

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!,$$

$$\epsilon_\lambda = \prod_{i=1}^{\infty} (-1)^{(i-1)m_i(\lambda)}.$$

It is easy to see that $\epsilon_\lambda = 1$ if a permutation with cycle type λ is even, and that $\epsilon_\lambda = -1$ if the permutation is odd.

Here are the theorems we prove in this section:

Theorem 1.1. For $n \in \mathbb{N}$, the following formula holds as an identity of formal power series of t:

$$\prod_{i=1}^{n} \frac{1}{1-t^i} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}}.$$

Theorem 1.2. For $n \in \mathbb{N}$, the following formula holds as an identity of formal power series of t:

$$\prod_{i=1}^{n} \frac{t^{i-1}}{1-t^i} = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}}.$$

It is also shown that the both sides of the equations of Theorem 1.1 and 1.2 are equal to the inner product of the elementary symmetric polynomials $\langle e_{(n)}, e_{(n)} \rangle_t$, and the inner product of the elementary symmetric polynomial and the complete symmetric polynomial $\langle e_{(n)}, h_{(n)} \rangle_t$, respectively.

1.1 Proof of Theorem 1.1 and 1.2 using symmetric polynomials

In this section we give a proof of Theorem 1.1 and 1.2 using symmetric polynomials. We fix a non-negative integer $n \in \mathbb{N}$. The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ by permuting the variables. A polynomial in $\mathbb{Q}[x_1, x_2, \dots, x_n]$ is called *symmetric* if it is invariant under this action. We denote the vector space of all symmetric polynomials in $\mathbb{Q}[x_1, x_2, \dots, x_n]$ by Λ_n.

Let $x^\alpha = {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots {x_n}^{\alpha_n}$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathscr{P}_n$, let $m_\lambda = m_\lambda(x_1, x_2, \dots, x_n) = \sum_\alpha x^\alpha$ where α runs over all distinct finite series $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ obtained by permuting the parts of the series of n non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_l, 0, \dots, 0)$. m_λ is called the *monomial symmetric polynomial* corresponding to λ. $\{m_\lambda \mid \lambda \in \mathscr{P}_n\}$ is a basis of Λ_n.

There are other well-known bases of Λ_n. First, the *elementary symmetric polynomial* $e_\lambda = e_\lambda(x_1, x_2, \dots, x_n)$ is defined by

$$e_\lambda = e_\lambda(x_1, x_2, \dots, x_n) = m_{\left(1^{\lambda_1}\right)} m_{\left(1^{\lambda_2}\right)} \cdots m_{\left(1^{\lambda_{l(\lambda)}}\right)}$$

for $\lambda \in \mathscr{P}'_n$. $\{e_\lambda \mid \lambda \in \mathscr{P}'_n\}$ is a basis of Λ_n [5, Theorem 5.3.5.]. Next, the *complete symmetric polynomial* $h_\lambda = h_\lambda(x_1, x_2, \ldots, x_n)$ is defined by

$$\begin{aligned} h_\lambda =& h_\lambda(x_1, x_2, \ldots, x_n) \\ =& \left(\sum_{\substack{\mu^1 \vdash \lambda_1 \\ l(\mu^1) \le n}} m_{\mu^1} \right) \left(\sum_{\substack{\mu^2 \vdash \lambda_2 \\ l(\mu^2) \le n}} m_{\mu^2} \right) \cdots \left(\sum_{\substack{\mu^{l(\lambda)} \vdash \lambda_{l(\lambda)} \\ l(\mu^{l(\lambda)}) \le n}} m_{\mu^{l(\lambda)}} \right) \end{aligned}$$

for $\lambda \in \mathscr{P}$. $\{h_\lambda \mid \lambda \in \mathscr{P}_n\}$ is a basis of Λ_n [5, Theorem 5.3.8.]. Finally, the *power sum symmetric polynomial* $p_\lambda = p_\lambda(x_1, x_2, \ldots, x_n)$ is defined by

$$p_\lambda = p_\lambda(x_1, x_2, \ldots, x_n) = m_{(\lambda_1)} m_{(\lambda_2)} \cdots m_{\left(\lambda_{l(\lambda)}\right)}$$

for $\lambda \in \mathscr{P}$. $\{p_\lambda \mid \lambda \in \mathscr{P}_n\}$ is a basis of Λ_n [5, Theorem 5.3.9.]. These bases of Λ_n have the following relations:

Proposition 1.3 ([4, Chapter I, (2.14')])**.** The following relations hold:

1. $e_{(n)} = \displaystyle\sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda$.

2. $h_{(n)} = \displaystyle\sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda$.

We also recall some concepts about the symmetric polynomials which have a parameter t to prove the theorems. Let $\Lambda_{t,n} = \Lambda_n \otimes \mathbb{Q}(t)$, which is the set of all elements in the polynomial ring $\mathbb{Q}(t)[x_1, x_2, \ldots, x_n]$ which are invariant under the permuting of the variables $x_1, x_2, \ldots, x_n$. We define a inner product $\langle \cdot, \cdot \rangle_t$ on $\Lambda_{t,n}$ by

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1}{1 - t^{\lambda_i}}$$

for $\lambda, \mu \in \mathscr{P}_n$, where $\delta_{\lambda\mu}$ is the Kronecker delta. For $\lambda \in \mathscr{P}_n$, the *Hall-Littlewood symmetric polynomial* $P_\lambda(t) = P_\lambda(x_1, x_2, \ldots, x_n; t) \in \Lambda_{t,n}$ is defined by

$$\begin{aligned} P_\lambda(t) =& P_\lambda(x_1, x_2, \ldots, x_n; t) \\ =& \left(\prod_{i \ge 0} \prod_{j=1}^{m_i(\lambda)} \frac{1-t}{1-t^j} \right) \sum_{w \in \mathfrak{S}_n} w \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right), \end{aligned}$$

where $m_0(\lambda) = n - l(\lambda)$. Especially if $\lambda = (1^n)$, $P_{(1^n)}(t) = e_{(n)}$ holds [4, Chapter III, (2.8)]. The inner product of the Hall-Littlewood symmetric polynomials satisfies the following property:

Proposition 1.4 ([4, Chapter III, (4.9)])**.** For $\lambda, \mu \in \mathscr{P}_n$, $\langle P_\lambda(t), Q_\mu(t) \rangle_t = \delta_{\lambda\mu}$ holds, where $Q_\mu(t) = \left(\displaystyle\prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\mu)} (1 - t^j) \right) P_\mu(t)$ for μ.

The complete symmetric polynomials can be expressed by the Hall-Littlewood symmetric polynomials as follows:

Proposition 1.5 ([4, Chapter III, 4, Example 1])**.** With $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$ for $\lambda \in \mathscr{P}$,

$$h_{(n)} = \sum_{\lambda\vdash n} t^{n(\lambda)} P_\lambda(t)$$

holds.

Now we can prove Theorem 1.1 and 1.2. By calculating the inner product $\langle e_{(n)}, e_{(n)}\rangle_t$, we obtain

$$\begin{aligned}
\langle e_{(n)}, e_{(n)}\rangle_t =& \left\langle \sum_{\lambda\vdash n} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda, \sum_{\mu\vdash n} \frac{\epsilon_\mu}{z_\mu} p_\mu \right\rangle_t = \sum_{\lambda,\mu\vdash n} \frac{\epsilon_\lambda \epsilon_\mu}{z_\lambda z_\mu} \langle p_\lambda, p_\mu\rangle_t \\
=& \sum_{\lambda\vdash n} \frac{{\epsilon_\lambda}^2}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}} = \sum_{\lambda\vdash n} \frac{1}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}}
\end{aligned}$$

and

$$\begin{aligned}
\langle e_{(n)}, e_{(n)}\rangle_t =& \langle P_{(1^n)}(t), P_{(1^n)}(t)\rangle_t = \left\langle P_{(1^n)}(t), \left(\prod_{i=1}^{n} \frac{1}{1-t^i}\right) Q_{(1^n)}(t) \right\rangle_t \\
=& \left(\prod_{i=1}^{n} \frac{1}{1-t^i}\right) \langle P_{(1^n)}(t), Q_{(1^n)}(t)\rangle_t = \prod_{i=1}^{n} \frac{1}{1-t^i},
\end{aligned}$$

which completes the proof of Theorem 1.1. Similarly, we obtain Theorem 1.2 by calculating $\langle e_{(n)}, h_{(n)}\rangle_t$ as follows:

$$\begin{aligned}
\langle e_{(n)}, h_{(n)}\rangle_t =& \left\langle \sum_{\lambda\vdash n} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda, \sum_{\mu\vdash n} \frac{1}{z_\mu} p_\mu \right\rangle_t = \sum_{\lambda,\mu\vdash n} \frac{\epsilon_\lambda}{z_\lambda z_\mu} \langle p_\lambda, p_\mu\rangle_t \\
=& \sum_{\lambda\vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}},
\end{aligned}$$

$$\begin{aligned}
\langle e_{(n)}, h_{(n)}\rangle_t =& \left\langle P_{(1^n)}(t), \sum_{\lambda\vdash n} t^{n(\lambda)} P_\lambda(t) \right\rangle_t = \sum_{\lambda\vdash n} t^{n(\lambda)} \langle P_{(1^n)}(t), P_\lambda(t)\rangle_t \\
=& \sum_{\lambda\vdash n} t^{n(\lambda)} \left\langle P_{(1^n)}(t), \left(\prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} \frac{1}{1-t^j}\right) Q_\lambda(t) \right\rangle_t \\
=& \sum_{\lambda\vdash n} t^{n(\lambda)} \left(\prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} \frac{1}{1-t^j}\right) \langle P_{(1^n)}(t), Q_\lambda(t)\rangle_t \\
=& t^{n(1^n)} \left(\prod_{i=1}^{n} \frac{1}{1-t^i}\right) = \prod_{i=1}^{n} \frac{t^{i-1}}{1-t^i}.
\end{aligned}$$

1.2 Bijective proof of Theorem 1.1 and 1.2

In this section we give another proof of Theorem 1.1 and 1.2 by constructing certain bijections. We fix a non-negative integer $n \in \mathbb{N}$.

First, we prove Theorem 1.1. Let

$$A_{n,d} = \left\{ (a_1, a_2, \ldots, a_n) \;\middle|\; a_i \in \mathbb{N}, \sum_{i=1}^{n} a_i \cdot i = d \right\}$$

for $d \in \mathbb{N}$,

$$B_{\lambda,d} = \left\{ (b_1, b_2 \ldots, b_{l(\lambda)}) \;\middle|\; b_i \in \mathbb{N}, \sum_{i=1}^{l(\lambda)} b_i \cdot \lambda_i = d \right\}$$

for $\lambda \vdash n$, $d \in \mathbb{N}$, and C_λ be the conjugacy class of $\mathfrak{S}_n$ corresponding to $\lambda \vdash n$. One obtains

$$n! \prod_{i=1}^{n} \frac{1}{1-t^i} = n! \prod_{i=1}^{n} (1 + t^i + t^{2i} + \cdots) = n! \sum_{d=0}^{\infty} |A_{n,d}| t^d = \sum_{d=0}^{\infty} |A_{n,d} \times \mathfrak{S}_n| t^d$$

and

$$\begin{aligned}\sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}} &= \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \prod_{i=1}^{l(\lambda)} (1 + t^{\lambda_i} + t^{2\lambda_i} + \cdots) = \sum_{\lambda \vdash n} |C_\lambda| \sum_{d=0}^{\infty} |B_{\lambda,d}| t^d \\ &= \sum_{\lambda \vdash n} \sum_{d=0}^{\infty} |B_{\lambda,d}||C_\lambda| t^d = \sum_{d=0}^{\infty} \left| \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda) \right| t^d.\end{aligned}$$

Therefore it suffices to construct a bijection

$$f_{n,d} : A_{n,d} \times \mathfrak{S}_n \to \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda)$$

for $d \in \mathbb{N}$.

To construct $f_{n,d}$, we define

$$f_{n,d}\big((a_1, a_2, \ldots, a_n), \sigma\big) \in \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda)$$

for $(a_1, a_2, \ldots, a_n) \in A_{n,d}$ and $\sigma \in \mathfrak{S}_n$ by the following algorithm:

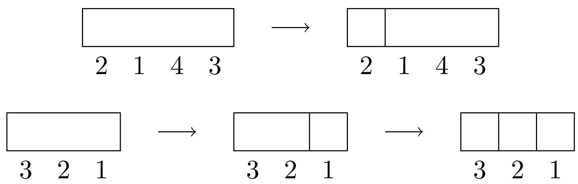
- **Step 1.** Draw the rim of the Young diagram of the partition

  $$(1^{a_1} 2^{a_2} \cdots n^{a_n}) \vdash d,$$

  and split it into blocks of columns by depth.

- **Step 2.** Write the numbers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ on each column from the left to the right. If the width of the diagram is less then n, add columns of depth 0 to the right of the diagram to make it has n columns before writing the numbers.

- **Step 3.** For each blocks of the diagram, split it at just to the left of the smallest number if the smallest number of the block is not at the most left place. Repeat the operation on new blocks until every block has its smallest number at the most left place. For example, the following diagrams need this operation once or twice, respectively.



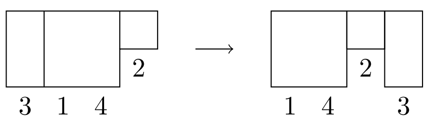
- **Step 4.** Rearrange the blocks by the following rules:
  1. Put wider one to the left.
  2. If there are blocks of the same width, put one which has the smallest number to the left.
- **Step 5.** Let l be the number of blocks in the diagram. Let λ_i be the width of the ith block from the left, and b_i be the depth of it for $1 \leq i \leq l$. It determines the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$. For $1 \leq i \leq l$, let τ_i be the cyclic permutation $(j_{i,1}, j_{i,2}, \dots, j_{i,\lambda_i}) \in \mathfrak{S}_n$ if ith block from the left has the numbers $j_{i,1}, j_{i,2}, \dots, j_{i,\lambda_i}$ from the left.
- **Step 6.** Define

$$f_{n,d}\big((a_1, a_2, \dots, a_n), \sigma\big) = \big((b_1, b_2, \dots, b_{l(\lambda)}), \tau_1\tau_2\cdots\tau_{l(\lambda)}\big) \in B_{\lambda,d} \times C_\lambda.$$

We illustrate this algorithm with an example

$$(0,0,1,1) \in A_{4,7} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \in \mathfrak{S}_4.$$

We get the following left diagram after Step 3, and the right one after Step 4.



Therefore we get

$$f_{4,7}\left((0,0,1,1), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right) = \big((2,1,2), (1,4)(2)(3)\big)$$
$$\in B_{(2,1,1),7} \times C_{(2,1,1)}.$$

We can define

$$f_{n,d}{}^{-1}\big((b_1, b_2, \dots, b_{l(\lambda)}), \tau\big) \in A_{n,d} \times \mathfrak{S}_n$$

for $\lambda \vdash n$, $(b_1, b_2, \dots, b_{l(\lambda)}) \in B_{\lambda,d}$ and $\tau \in C_\lambda$ to construct the inverse function

$$f_{n,d}{}^{-1} : \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda) \to A_{n,d} \times \mathfrak{S}_n$$

by the following algorithm:

- **Step 1.** Suppose

$$\tau = (r_{1,1}, r_{1,2}, \ldots, r_{1,\lambda_1})(r_{2,1}, r_{2,2}, \ldots, r_{2,\lambda_2}) \cdots (r_{l(\lambda),1}, r_{l(\lambda),2}, \ldots, r_{l(\lambda),\lambda_{l(\lambda)}})$$

is the decomposition of τ into disjoint cycles with conditions

$$r_{i,1} = \min\{r_{i,1}, r_{i,2}, \ldots, r_{i,\lambda_i}\}$$

for each i, and $r_{i,1} < r_{i+1,1}$ if $\lambda_i = \lambda_{i+1}$.

- **Step 2.** For $1 \le i \le l(\lambda)$, let X_i be the block of width λ_i, depth b_i. Write the numbers $r_{i,1}, r_{i,2}, \ldots, r_{i,\lambda_i}$ on each column of X_i from the left to the right.

- **Step 3.** Arrange the blocks $X_1, X_2, \ldots, X_{l(\lambda)}$ by the following rules:

  1. Put deeper one to the left.
  2. If there are blocks of the same depth, put one which has the smallest number to the right.

- **Step 4.** Define ${f_{n,d}}^{-1}\big((b_1, b_2, \ldots, b_{l(\lambda)}), \tau\big) = \big((a_1, a_2, \ldots, a_n), \sigma\big)$ where the blocks form the Young diagram of shape $(1^{a_1}2^{a_2}\cdots n^{a_n})$, and each column of the diagram has the numbers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ from the left.

By the above two algorithms, we can see that $f_{n,d}$ is a bijection between $A_{n,d} \times \mathfrak{S}_n$ and $\displaystyle\bigsqcup_{\lambda \vdash n}(B_{\lambda,d} \times C_\lambda)$. It completes the proof of Theorem 1.1.

Next, we prove Theorem 1.2. We call $\lambda \in \mathscr{P}$ *even* if $\epsilon_\lambda = 1$, and we call it *odd* if $\epsilon_\lambda = -1$. One obtains

$$\begin{aligned} n! \prod_{i=1}^{n} \frac{t^{i-1}}{1-t^i} &= t^{\binom{n}{2}} \cdot n! \prod_{i=1}^{n} \frac{1}{1-t^i} = t^{\binom{n}{2}} \cdot \sum_{d=0}^{\infty} |A_{n,d} \times \mathfrak{S}_n| t^d \\ &= \sum_{d=0}^{\infty} \left| A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n \right| t^d \end{aligned}$$

and

$$\begin{aligned} n! \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}} &= \sum_{\lambda \vdash n} \epsilon_\lambda \frac{n!}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}} = \sum_{\lambda \vdash n} \epsilon_\lambda \sum_{d=0}^{\infty} |B_{\lambda,d} \times C_\lambda| t^d \\ &= \sum_{d=0}^{\infty} \left(\sum_{\lambda \vdash n} \epsilon_\lambda |B_{\lambda,d} \times C_\lambda| \right) t^d \\ &= \sum_{d=0}^{\infty} \left(\sum_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} |B_{\lambda,d} \times C_\lambda| - \sum_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} |B_{\lambda,d} \times C_\lambda| \right) t^d \\ &= \sum_{d=0}^{\infty} \left(\left| \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda) \right| - \left| \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda) \right| \right) t^d. \end{aligned}$$

Therefore it suffices to construct a bijection

$$g_{n,d} : \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda) \right) \sqcup \left(A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n \right) \to \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda)$$

for $d \in \mathbb{N}$. We construct $g_{n,d}$ by constructing two bijections

$$g_{n,d,1} : \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda) \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda) \right) \setminus \mathrm{AD}_{n,d},$$

$$g_{n,d,2} : A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n \to \mathrm{AD}_{n,d},$$

where $\mathrm{AD}_{n,d}$ is the set of all elements

$$\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big)$$

in $B_{(1^n),d} \times C_{(1^n)}$ such that $b_1, b_2, \ldots, b_n$ are all distinct.

We construct $g_{n,d,1}$ by the involution I on $\left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda) \right) \setminus \mathrm{AD}_{n,d}$ defined by the following algorithm:

- **Step 1.** Take $\mu \vdash n$ and $\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in (B_{\mu,d} \times C_\mu) \setminus \mathrm{AD}_{n,d}$.
- **Step 2.** Take

  $$\big((a_1, a_2, \ldots, a_n), \sigma\big) = {f_{n,d}}^{-1}\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in A_{n,d} \times \mathfrak{S}_n,$$

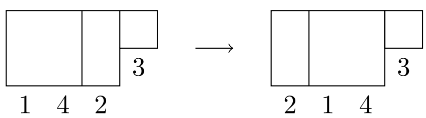
  where $f_{n,d}$ is the one constructed to prove Theorem 1.1.
- **Step 3.** Let $J = \{j \in \{1, 2, \ldots, n-1\} \mid a_j = 0\}$. J is not empty because $\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big)$ is not in $\mathrm{AD}_{n,d}$. Hence we can take $j_0 = \min J$.
- **Step 4.** Define

  $$\begin{aligned} I\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) =& f_{n,d}\big((a_1, a_2, \ldots, a_n), \sigma \cdot (j_0, j_0 + 1)\big) \\ &\in \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times C_\lambda). \end{aligned}$$

We illustrate this algorithm with an example

$$(2, 2, 1) \in B_{(2,1,1),7} \quad \text{and} \quad (1, 4)(2)(3) \in C_{(2,1,1)}.$$

By using the algorithm for ${f_{n,d}}^{-1}$ in Theorem 1.1, we obtain the following diagram.



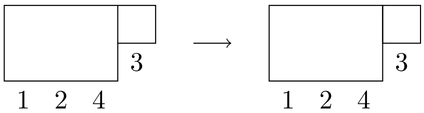
Thus we get

$$f_{4,7}{}^{-1}\big((2,2,1),(1,4)(2)(3)\big) = \left((0,0,1,1), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\right) \in A_{4,7} \times \mathfrak{S}_4$$

in Step 1. In the case, $J = \{1,2\}$ and $j_0 = 1$. Therefore in Step 4, we get

$$\begin{aligned} I\big((2,2,1),(1,4)(2)(3)\big) =& f_{4,7}\left((0,0,1,1), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot (1,2)\right) \\ =& f_{4,7}\left((0,0,1,1), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\right). \end{aligned}$$

By the algorithm for $f_{n,d}$ in Theorem 1.1, we obtain the following diagram from $(0,0,1,1) \in A_{4,7}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \in \mathfrak{S}_4$.



Therefore

$$\begin{aligned} I\big((2,2,1),(1,4)(2)(3)\big) =& f_{4,7}\left((0,0,1,1), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\right) \\ =& \big((2,1),(1,2,4)(3)\big) \\ \in& B_{(3,1),7} \times C_{(3,1)} \\ \subseteq& \bigsqcup_{\substack{\lambda \vdash 4 \\ \lambda:\text{even}}} (B_{\lambda,7} \times C_\lambda). \end{aligned}$$

The operation of (j_0, j_0+1) on σ in the Step 4 exchanges the numbers written on the most left two column of the same depth. Hence it exchanges even number of blocks of the depth to odd number of them and vise versa. Moreover, the algorithm does not give an element of $\mathrm{AD}_{n,d}$ because it does not change the depth of each column of the diagram. Therefore one can see that

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda)\right) \setminus \mathrm{AD}_{n,d}$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda)$, and

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda)$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), \tau\big) \in \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda)\right) \setminus \mathrm{AD}_{n,d}$. Since I is involution, we obtain a bijection

$$g_{n,d,1} : \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda) \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda) \right) \setminus \mathrm{AD}_{n,d}.$$

Next, we define

$$g_{n,d,2}\big((a_1, a_2, \ldots, a_n), \sigma\big) \in \mathrm{AD}_{n,d}$$

for $(a_1, a_2, \ldots, a_n) \in A_{n,d-\binom{n}{2}}$ and $\sigma \in \mathfrak{S}_n$ by

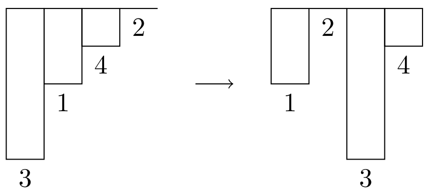
$$g_{n,d,2}\big((a_1, a_2, \ldots, a_n), \sigma\big) = f_{n,d}\big((a_1+1, a_2+1, \ldots, a_{n-1}+1, a_n), \sigma\big),$$

where $f_{n,d}$ is the one constructed to prove Theorem 1.1.

For example, for $(1,0,0,0) \in A_{4,1}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, we have

$$g_{4,7,2}\left((1,0,0,0), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right) = f_{4,7}\left((2,1,1,0), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right).$$

Using the algorithm for $f_{4,7}$, we obtain the following diagrams for $(2,1,1,0) \in A_{4,7}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \in \mathfrak{S}_4$.



Therefore, we have

$$\begin{aligned} g_{4,7}\left((1,0,0,0), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right) =& f_{4,7}\left((2,1,1,0), \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}\right) \\ =& \big((2,0,4,1), (1)(2)(3)(4)\big) \\ \in& B_{(1,1,1,1),7} \times C_{(1,1,1,1)}. \end{aligned}$$

Since the diagram made from $\big((a_1+1, a_2+1, \ldots, a_{n-1}+1, a_n), \sigma\big)$ by the algorithm for $f_{n,d}$ has columns of all distinct depths, $g_{n,d,2}\big((a_1, a_2, \ldots, a_n), \sigma\big)$ is an element of $\mathrm{AD}_{n,d}$.

We can construct the inverse function

$$g_{n,d,2}{}^{-1} : \mathrm{AD}_{n,d} \to A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n$$

by defining

$$g_{n,d,2}{}^{-1}\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big) \in A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n$$

for $\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big) \in \mathrm{AD}_{n,d}$ by the following algorithm:

- **Step 1.** Take

  $$\big((a_1, a_2, \ldots, a_n), \sigma\big) = f_{n,d}{}^{-1}\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big) \in A_{n,d} \times \mathfrak{S}_n,$$

  where $f_{n,d}$ is the one constructed to prove Theorem 1.1.

- **Step 2.** Define

  $$\begin{aligned} &g_{n,d,2}\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big) \\ =&\big((a_1 - 1, a_2 - 1, \ldots, a_{n-1} - 1, a_n), \sigma\big) \\ \in&A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n. \end{aligned}$$

  Since $\big((b_1, b_2, \ldots, b_n), (1)(2)\cdots(n)\big) \in \mathrm{AD}_{n,d},\ a_1, a_2, \ldots, a_{n-1} \neq 0$ and $(a_1 - 1, a_2 - 1, \ldots, a_{n-1} - 1, a_n) \in A_{n,d-\binom{n}{2}}$.

Therefore we have defined a bijection

$$g_{n,d,2} : A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n \to \mathrm{AD}_{n,d}.$$

By two bijections $g_{n,d,1}$ and $g_{n,d,2}$ we have constructed, now we have a bijection

$$g_{n,d} : \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times C_\lambda) \right) \sqcup \left(A_{n,d-\binom{n}{2}} \times \mathfrak{S}_n \right) \to \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times C_\lambda)$$

and it completes the proof of Theorem 1.2.

1.3 Generalizations of Theorem 1.1 and 1.2

In Section 1.1, we gave a proof of Theorem 1.1 by calculating the inner product

$$\langle e_{(n)}, e_{(n)} \rangle_t = \langle P_{(1^n)}(t), P_{(1^n)}(t) \rangle_t.$$

We can easily generalize it to the inner product $\langle P_\lambda(t), P_\lambda(t) \rangle_t$ of an arbitrary Hall-Littlewood polynomial as follows using Theorem 1.1 and Proposition 1.4.

Theorem 1.6. For $\lambda \in \mathscr{P}$, the following formula holds as an identity of formal power series of t:

$$\langle P_\lambda(t), P_\lambda(t) \rangle_t = \prod_{j=1}^{\infty} \prod_{i=1}^{m_j(\lambda)} \frac{1}{1 - t^i} = \prod_{j=1}^{\infty} \sum_{\mu \vdash m_j(\lambda)} \frac{1}{z_\mu} \prod_{i=1}^{l(\mu)} \frac{1}{1 - t^{\mu_i}}.$$

We give another generalization of Theorem 1.1 and Theorem 1.2. The *Schur polynomial* s_λ is a symmetric polynomial defined by

$$s_\lambda = s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \le i,j \le n}}{\det\left(x_i^{n-j}\right)_{1 \le i,j \le n}}$$

for $\lambda \in \mathscr{P}_n$. $\{s_\lambda \mid \lambda \in \mathscr{P}_n\}$ is a basis of Λ_n [5, Theorem 5.4.4.]. The *Kostka polynomial* $K_{\lambda\mu}(t) \in \mathbb{Q}(t)$ corresponding to $\lambda, \mu \in \mathscr{P}$ is defined as the entry of the transition matrix from the basis of Schur polynomials and the basis of Hall-Littlewood polynomials:

$$s_\lambda = \sum_{\mu \in \mathscr{P}} K_{\lambda\mu}(t) P_\mu(t) \quad (\lambda \in \mathscr{P}).$$

Schur polynomials and power sum polynomials enjoy the following relation:

Proposition 1.7 ([4, Chapter I, p114])**.** For $\lambda \vdash n$, $s_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu$, where χ^λ is the irreducible character of $\mathfrak{S}_n$ corresponding to λ.

By calculating the inner product $\langle s_\lambda, s_\mu \rangle_t$, we show the following theorem:

Theorem 1.8. For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathscr{P}$, the following formula holds as an identity of formal power series of t:

$$\langle s_\lambda, s_\mu \rangle_t = \sum_{\nu \vdash n} K_{\lambda\nu}(t) K_{\mu\nu}(t) \prod_{j=1}^{\infty} \prod_{i=1}^{m_j(\nu)} \frac{1}{1-t^i} = \sum_{\nu \vdash n} \frac{\chi^\lambda(\nu)\chi^\mu(\nu)}{z_\nu} \prod_{i=1}^{l(\nu)} \frac{1}{1-t^{\nu_i}}.$$

Since $s_{(1^n)} = e_{(n)}$ and $s_{(n)} = h_{(n)}$ [4, Chapter I, (3.9)], Theorem 1.8 is a generalization of Theorem 1.1 and 1.2. Now we prove Theorem 1.8. For $n \in \mathbb{N}$ and $\lambda \vdash n$, we define $S_\lambda(t) = S_\lambda(x_1, x_2, \ldots, x_n; t) \in \Lambda_{t,n}$ by

$$S_\lambda(t) = S_\lambda(x_1, x_2, \ldots, x_n; t) = \det\left(q_{\lambda_i - i + j}(t)\right)_{1 \le i,j \le l(\lambda)},$$

where $q_r(t) = Q_{(r)}(t) = (1-t)P_{(r)}(t)$ for $r \geq 1$, $q_0(t) = 1$, and $q_r(t) = 0$ if $r < 0$. $S_\lambda(t)$ and Schur polynomials have the following properties:

Proposition 1.9 ([4, Chapter III, p241])**.** For $\lambda, \mu \vdash n$,

$$s_\lambda = \sum_{\mu \vdash n} \left(\sum_{\nu \vdash n} K_{\lambda\nu}(t) K_{\mu\nu}(t) \prod_{j=1}^{\infty} \prod_{i=1}^{m_j(\nu)} \frac{1}{1-t^i} \right) S_\mu(t).$$

Proposition 1.10 ([4, Chapter III, (4.10)])**.** For $\lambda, \mu \vdash n$, $\langle S_\lambda(t), s_\mu \rangle_t = \delta_{\lambda\mu}$ holds.

Using these propositions, we can calculate

$$\begin{aligned}
\langle s_\lambda, s_\mu \rangle_t &= \left\langle \sum_{\rho \vdash n} \frac{\chi^\lambda(\rho)}{z_\rho} p_\rho, \sum_{\sigma \vdash n} \frac{\chi^\mu(\sigma)}{z_\sigma} p_\sigma \right\rangle_t = \sum_{\rho,\sigma \vdash n} \frac{\chi^\lambda(\rho)\chi^\mu(\sigma)}{z_\rho z_\sigma} \langle p_\rho, p_\sigma \rangle_t \\
&= \sum_{\nu \vdash n} \frac{\chi^\lambda(\nu)\chi^\mu(\nu)}{z_\nu} \prod_{i=1}^{l(\nu)} \frac{1}{1-t^{\nu_i}}
\end{aligned}$$

and

$$\begin{aligned}
\langle s_\lambda, s_\mu \rangle_t &= \left\langle \sum_{\rho \vdash n} \left(\sum_{\nu \vdash n} K_{\lambda\nu}(t) K_{\rho\nu}(t) \prod_{j=1}^{\infty} \prod_{i=1}^{m_j(\nu)} \frac{1}{1-t^i} \right) S_\rho(t), s_\mu \right\rangle_t \\
&= \sum_{\nu \vdash n} K_{\lambda\nu}(t) K_{\mu\nu}(t) \prod_{j=1}^{\infty} \prod_{i=1}^{m_j(\nu)} \frac{1}{1-t^i},
\end{aligned}$$

which completes the proof of Theorem 1.8.

We do not know a bijective proof of Theorem 1.8. It would be an interesting problem to prove Theorem 1.8 bijectively like Theorem 1.1 and 1.2.

2 The case of two parameters

In this section we prove the following identities of two parameters q, t as generalizations of Theorem 1.1 and 1.2.

Theorem 2.1. For $n \in \mathbb{N}$, the following formula holds as an identity of formal power series of q, t:

$$\prod_{i=1}^{n} \frac{1 - qt^{i-1}}{1 - t^i} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

Theorem 2.2. For $n \in \mathbb{N}$, the following formula holds as an identity of formal power series of q, t:

$$\prod_{i=1}^{n} \frac{t^{i-1} - q}{1 - t^i} = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

It is also shown that the both sides of the equations of Theorem 2.1 and 2.2 are equal to the q, t-inner product $\langle e_{(n)}, e_{(n)} \rangle_{q,t}$ and $\langle e_{(n)}, h_{(n)} \rangle_{q,t}$, respectively.

2.1 A proof of Theorem 2.1 and 2.2 using symmetric polynomials

In this section we prove Theorem 2.1 and 2.2 using symmetric polynomials. Let $\Lambda_{q,t,n} = \Lambda_n \otimes \mathbb{Q}(q,t)$, which is the vector space of all elements in the polynomial ring $\mathbb{Q}(q,t)[x_1, x_2, \ldots, x_n]$ which are invariant under the permutations of the variables $x_1, x_2, \ldots, x_n$. We define a inner product $\langle \cdot, \cdot \rangle_{q,t}$ of $\Lambda_{q,t,n}$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

for $\lambda, \mu \in \mathscr{P}_n$. We also define the partial order on $\mathscr{P}(n)$ called *dominance order* by

$$\mu \leq \lambda \iff \forall i \in \{1, 2, \ldots, n\}, \mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for $\lambda, \mu \in \mathscr{P}(n)$, where $\lambda_j = 0$ for $l(\lambda) < j$, $\mu_j = 0$ for $l(\mu) < j$. The following proposition holds for the inner product and the partial order:

Proposition 2.3 ([4, Chapter VI, (4.7)])**.** There is a unique family

$$\{P_\lambda(q,t)\}_{\lambda \vdash n}$$

which consists of elements of $\Lambda_{q,t,n}$ satisfying the following conditions:

1. There is a map

$$\begin{array}{rccc} u: & \{(\lambda, \mu) \mid \lambda, \mu \in \mathscr{P}(n), \mu \leq \lambda\} & \to & \mathbb{Q}(q,t) \\ & \Psi & & \Psi \\ & (\lambda, \mu) & \mapsto & u_{\lambda\mu} \end{array}$$

   satisfying the following:

(a) For $\lambda \in \mathscr{P}(n)$, $P_\lambda(q,t) = \sum_{\substack{\mu \vdash n \\ \mu \le \lambda}} u_{\lambda\mu} m_\mu$.

(b) For $\lambda \in \mathscr{P}(n)$, $u_{\lambda\lambda} = 1$.

2. For $\lambda, \mu \in \mathscr{P}(n)$ such that $\lambda \neq \mu$, $\langle P_\lambda(q,t), P_\mu(q,t)\rangle_{q,t} = 0$.

The symmetric polynomials $P_\lambda(q,t)$ defined by the proposition are called the *Macdonald polynomials*. In particular, $P_{(1^n)}(q,t) = e_{(n)}$ [4, Chapter VI, (4.8)]. Let $b_\lambda(q,t) = \langle P_\lambda(q,t), P_\lambda(q,t)\rangle_{q,t}^{-1}$ and $Q_\lambda(q,t) = b_\lambda(q,t)P_\lambda(q,t)$, so $\langle P_\lambda(q,t), Q_\mu(q,t)\rangle_{q,t} = \delta_{\lambda\mu}$. The following explicit formula of $b_\lambda(q,t)$ is known:

Proposition 2.4 ([4, Chapter VI, (6.19)])**.** For $\lambda \in \mathscr{P}$,

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1 - q^{a(s)}t^{l(s)+1}}{1 - q^{a(s)+1}t^{l(s)}}$$

holds, where the right hand side is the product for all cells s in the Young diagram of λ, and for a cell s, $a(s)$ is the number of cells right of s in the same row as s in the diagram, and $l(s)$ is the number of cells below s in the same column as s in the diagram.

By calculating the inner product $\langle e_{(n)}, e_{(n)}\rangle_{q,t}$, we obtain

$$\begin{aligned}\langle e_{(n)}, e_{(n)}\rangle_{q,t} =& \left\langle \sum_{\lambda\vdash n} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda, \sum_{\mu\vdash n} \frac{\epsilon_\mu}{z_\mu} p_\mu \right\rangle_{q,t} = \sum_{\lambda,\mu\vdash n} \frac{\epsilon_\lambda\epsilon_\mu}{z_\lambda z_\mu} \langle p_\lambda, p_\mu\rangle_{q,t} \\ =& \sum_{\lambda\vdash n} \frac{1}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\end{aligned}$$

and

$$\begin{aligned}\langle e_{(n)}, e_{(n)}\rangle_{q,t} =& \langle P_{(1^n)}(q,t), P_{(1^n)}(q,t)\rangle_{q,t} = b_{(1^n)}(q,t)^{-1} \\ =& \prod_{s\in(1^n)} \frac{1 - q^{a(s)+1}t^{l(s)}}{1 - q^{a(s)}t^{l(s)+1}} = \prod_{i=1}^{n} \frac{1-qt^{n-i}}{1-t^{n-i+1}} = \prod_{i=1}^{n} \frac{1-qt^{i-1}}{1-t^i},\end{aligned}$$

which completes the proof of Theorem 2.1.

Next, we prove Theorem 2.2. Let $\Lambda_{q,t} = \bigoplus_{n\ge 0} \Lambda_{q,t,n}$ and $\varepsilon_{q,t} : \Lambda_{q,t} \to \mathbb{Q}(q,t)$ be the homomorphism defined by

$$\varepsilon_{q,t}(p_{(r)}) = \frac{1-q^r}{1-t^r}.$$

The followings are known about $P_\lambda(q,t)$, $Q_\lambda(q,t)$ and $\varepsilon_{q,t}$:

Proposition 2.5 ([4, Chapter VI, (2.6) and (4.13)])**.** For $n \in \mathbb{N}$,

$$\sum_{\lambda\vdash n} P_\lambda(x;q,t) Q_\lambda(y;q,t) = \sum_{\mu\vdash n} \frac{1}{z_\mu} \left(\prod_{i=1}^{l(\mu)} \frac{1-t^{\mu_i}}{1-q^{\mu_i}} \right) p_\mu(x) p_\mu(y)$$

holds, where $P_\lambda(x;q,t), p_\mu(x)$ are symmetric polynomials of variables $x_1, x_2, \ldots, x_n$, and $Q_\lambda(y;q,t), p_\mu(y)$ are symmetric polynomials of variables $y_1, y_2, \ldots, y_n$.

Proposition 2.6 ([4, Chapter VI, (6.17)])**.** For $\lambda \in \mathscr{P}$,

$$\varepsilon_{q,t}\big(P_\lambda(q,t)\big) = \prod_{s\in\lambda} \frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)}t^{l(s)+1}}$$

holds, where $a'(s)$ is the number of cells left of s in the same row as s in the diagram, and $l'(s)$ is the number of cells above s in the same column as s in the diagram for a cell s.

Since one obtains

$$\begin{aligned}\sum_{\lambda\vdash n} P_\lambda(x;q,t)\varepsilon_{q,t}\big(Q_\lambda(y;q,t)\big) &= \sum_{\lambda\vdash n} b_\lambda(q,t)P_\lambda(x;q,t)\varepsilon_{q,t}\big(P_\lambda(y;q,t)\big) \\ &= \sum_{\lambda\vdash n} b_\lambda(q,t)\left(\prod_{s\in\lambda}\frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)}t^{l(s)+1}}\right) P_\lambda(x;q,t) \\ &= \sum_{\lambda\vdash n}\left(\prod_{s\in\lambda}\frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)+1}t^{l(s)}}\right) P_\lambda(x;q,t)\end{aligned}$$

and

$$\begin{aligned}&\sum_{\mu\vdash n}\frac{1}{z_\mu}\left(\prod_{i=1}^{l(\mu)}\frac{1-t^{\mu_i}}{1-q^{\mu_i}}\right) p_\mu(x)\varepsilon_{q,t}\big(p_\mu(y)\big) \\ =&\sum_{\mu\vdash n}\frac{1}{z_\mu}\left(\prod_{i=1}^{l(\mu)}\frac{1-t^{\mu_i}}{1-q^{\mu_i}}\right) p_\mu(x)\left(\prod_{i=1}^{l(\mu)}\frac{1-q^{\mu_i}}{1-t^{\mu_i}}\right) \\ =&\sum_{\mu\vdash n}\frac{1}{z_\mu}p_\mu(x) = h_{(n)}(x)\end{aligned}$$

by applying $\varepsilon_{q,t}$ with respect to variables $y_1, y_2, \ldots, y_n$,

$$h_{(n)} = \sum_{\lambda\vdash n}\left(\prod_{s\in\lambda}\frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)+1}t^{l(s)}}\right) P_\lambda(q,t)$$

holds by Proposition 2.5. Now we complete the proof of Theorem 2.2 by calculating $\langle e_{(n)}, h_{(n)}\rangle_{q,t}$ as follows:

$$\begin{aligned}\langle e_{(n)}, h_{(n)}\rangle_{q,t} = \left\langle \sum_{\lambda\vdash n}\frac{\epsilon_\lambda}{z_\lambda}p_\lambda, \sum_{\mu\vdash n}\frac{1}{z_\mu}p_\mu\right\rangle_{q,t} &= \sum_{\lambda,\mu\vdash n}\frac{\epsilon_\lambda}{z_\lambda z_\mu}\langle p_\lambda, p_\mu\rangle_{q,t} \\ =\sum_{\lambda\vdash n}\frac{\epsilon_\lambda}{z_\lambda}\prod_{i=1}^{l(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}},\end{aligned}$$

$$
\begin{aligned}
\langle e_{(n)}, h_{(n)} \rangle_{q,t} &= \left\langle P_{(1^n)}(q,t), \sum_{\mu \vdash n} \left(\prod_{s \in \mu} \frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} \right) P_\mu(q,t) \right\rangle_{q,t} \\
&= \left(\prod_{s \in (1^n)} \frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} \right) \frac{1}{b_{(1^n)}(q,t)} \\
&= \prod_{s \in (1^n)} \frac{t^{l'(s)} - q^{a'(s)+1}}{1 - q^{a(s)} t^{l(s)+1}} \\
&= \prod_{i=1}^{n} \frac{t^{i-1} - q}{1 - t^i}.
\end{aligned}
$$

2.2 A bijective proof of Theorem 2.1 and 2.2

In this section we prove Theorem 2.1 and 2.2 by constructing certain bijections. We fix a non-negative integer $n \in \mathbb{N}$.

First, we prove Theorem 2.1. Let

$$
D_{n,d,e} = \left\{ J \subseteq \{0, 1, \ldots, n-1\} \;\middle|\; \sum_{j \in J} j = d, |J| = e \right\}
$$

for $e, k \in \mathbb{N}$. One obtains

$$
\prod_{i=1}^{n} (1 - qt^{i-1}) = \sum_{d=0}^{\infty} \sum_{e=0}^{n} |D_{n,d,e}| t^d (-q)^e,
$$

and therefore

$$
\begin{aligned}
n! \prod_{i=1}^{n} \frac{1 - qt^{i-1}}{1 - t^i} &= n! \prod_{i=1}^{n} (1 + t^i + t^{2i} + \cdots) \cdot \prod_{i=1}^{n} (1 - qt^{i-1}) \\
&= |\mathfrak{S}_n| \sum_{d_1=0}^{\infty} |A_{n,d_1}| t^{d_1} \cdot \sum_{d_2=0}^{\infty} \sum_{e=0}^{n} |D_{n,d_2,e}| t^{d_2} (-q)^e \\
&= \sum_{d=0}^{\infty} \sum_{e=0}^{n} (-1)^e \left| \bigsqcup_{d_1 + d_2 = d} (A_{n,d_1} \times D_{n,d_2,e} \times \mathfrak{S}_n) \right| t^d q^e.
\end{aligned}
$$

Next, let

$$
\begin{aligned}
E^+_{\lambda,e} &= \left\{ J \subseteq \{1, 2, \ldots, l(\lambda)\} \;\middle|\; \sum_{j \in J} \lambda_j = e, |J| \text{ is even} \right\}, \\
E^-_{\lambda,e} &= \left\{ J \subseteq \{1, 2, \ldots, l(\lambda)\} \;\middle|\; \sum_{j \in J} \lambda_j = e, |J| \text{ is odd} \right\},
\end{aligned}
$$

and then we obtain

$$\begin{aligned}
&\sum_{\lambda\vdash n}\frac{n!}{z_\lambda}\prod_{i=1}^{l(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\\
&=\sum_{\lambda\vdash n}\frac{n!}{z_\lambda}\left(\prod_{i=1}^{l(\lambda)}(1+t^{\lambda_i}+t^{2\lambda_i}+\cdots)\cdot\prod_{i=1}^{l(\lambda)}(1-q^{\lambda_i})\right)\\
&=\sum_{\lambda\vdash n}|C_\lambda|\left(\sum_{d=0}^{\infty}|B_{\lambda,d}|t^d\cdot\sum_{e=0}^{n}\left(\left|E^+_{\lambda,e}\right|-\left|E^-_{\lambda,e}\right|\right)q^e\right)\\
&=\sum_{d=0}^{\infty}\sum_{e=0}^{n}\sum_{\lambda\vdash n}\left(|B_{\lambda,d}|\left|E^+_{\lambda,e}\right||C_\lambda|-|B_{\lambda,d}|\left|E^-_{\lambda,e}\right||C_\lambda|\right)t^dq^e\\
&=\sum_{d=0}^{\infty}\sum_{e=0}^{n}\left(\left|\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)\right|-\left|\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^-_{\lambda,e}\times C_\lambda)\right|\right)t^dq^e.
\end{aligned}$$

Therefore it suffices to construct a bijection

$$\begin{aligned}
f_{n,d,e}:&\left(\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^-_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,e}\times\mathfrak{S}_n)\right)\\
&\to\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)
\end{aligned}$$

for $d\in\mathbb{N}$ and even number e satisfying $0\le e\le n$, and

$$\begin{aligned}
f_{n,d,e}:&\left(\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,e}\times\mathfrak{S}_n)\right)\\
&\to\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^-_{\lambda,e}\times C_\lambda)
\end{aligned}$$

for $d\in\mathbb{N}$ and odd number e satisfying $0\le e\le n$. Here we construct a bijection $f_{n,d,e}$ for $d\in\mathbb{N}$ and even number e satisfying $0\le e\le n$. We construct $f_{n,d,e}$ by constructing two bijections

$$f_{n,d,e,1}:\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^-_{\lambda,e}\times C_\lambda)\to\left(\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)\right)\setminus\mathrm{AD}_{n,d,e},$$

$$f_{n,d,e,2}:\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,e}\times\mathfrak{S}_n)\to\mathrm{AD}_{n,d,e},$$

where $\mathrm{AD}_{n,d,e}$ is the set of all elements

$$\big((b_1,b_2,\dots,b_{l(\lambda)}),J,\tau\big)$$

in $\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)$ such that $\lambda_j=1$ for all $j\in J$, and b_j for $j\in J$ are all distinct.

We construct $f_{n,d,e,1}$ by the involution I on

$$\left(\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^-_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{\lambda\vdash n}(B_{\lambda,d}\times E^+_{\lambda,e}\times C_\lambda)\right)\setminus\mathrm{AD}_{n,d,e}$$

defined by the following algorithm:

- **Step 1.** Take $\mu \vdash n$ and
$$\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in (B_{\mu,d} \times E^-_{\mu,e} \times C_\mu) \sqcup (B_{\mu,d} \times E^+_{\mu,e} \times C_\mu)$$
satisfying $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}_{n,d,e}$.

- **Step 2.** Use Step 1 and 2 of the algorithm for ${f_{n,d}}^{-1}$ in Theorem 1.1, and obtain a diagram of blocks. The ith block from the left is of width μ_i and depth b_i, and each column has number defined by τ.

- **Step 3.** For $j \in J$, paint the jth block from the left. There are e painted columns because $J \in E^+_{\mu,e} \sqcup E^-_{\mu,e}$.

- **Step 4.** Rearrange the blocks by the following rules:
  1. Put deeper one to the left.
  2. If there are blocks of the same depth, put painted one to the left.
  3. If there are blocks of the same depth and painting, put one which has the smallest number to the left.

- **Step 5.** Since $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}_{n,d,e}$, there is a painted block with width at least 2, or there are painted blocks of width 1 and same depth. Therefore there is a depth δ such that there are at least two painted columns of depth δ. Let δ_0 be the deepest such depth δ.

- **Step 6.** Write the painted blocks of depth δ_0 $X_1, X_2, \ldots$ from left. For each j, write the numbers written on each column of X_j $q_{j,1}, q_{j,2}, \ldots$ from left. Let
$$\rho = (q_{1,1}, q_{1,2}, \ldots)(q_{2,1}, q_{2,2}, \ldots) \cdots \in \mathfrak{S}_n,$$
and i_1, i_2 be the smallest two numbers in
$$\{q_{1,1}, q_{1,2}, \ldots\} \cup \{q_{2,1}, q_{2,2}, \ldots\} \cup \cdots$$
which satisfies $i_1 < i_2$.

- **Step 7.** $(i_1, i_2)\rho$ is written $(r_{1,1}, r_{1,2}, \ldots)(r_{2,1}, r_{2,2}, \ldots) \cdots$ by some disjoint cycles $(r_{j,1}, r_{j,2}, \ldots)$ satisfying $r_{j,1} = \min\{r_{j,1}, r_{j,2}, \ldots\}$ for each j, and $r_{1,1} < r_{2,1} < \cdots$. For each j, let Y_j be the painted block whose width is the length of $(r_{j,1}, r_{j,2}, \ldots)$ and whose depth is δ_0, and whose each column has the numbers $r_{j,1}, r_{j,2}, \ldots$ from the left. Arrange $Y_1, Y_2, \ldots$ from the left, and exchange the all painted blocks of depth δ_0 of the diagram made in Step 4 for them.

  The operation of (i_1, i_2) on ρ changes the sign of ρ. Hence $(i_1, i_2)\rho$ consists of even numbers of cycles if ρ consists of odd numbers of them, and vice versa. Therefore this step exchanges the parity of the number of painted blocks in the diagram.

- **Step 8.** Use Step 4 to Step 6 of the algorithm for $f_{n,d}$ in Theorem 1.1, and take the corresponding partition $\lambda \vdash n$ and the elements of $B_{\lambda,d}$ and C_λ.

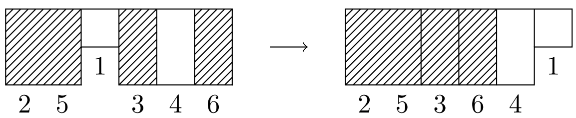
- **Step 9.** Put J' be the subset of $\{1, 2, \dots, l(\lambda)\}$ such that $j \in J'$ if and only if the jth block from the left of the diagram is painted. One can see $J' \in E^+_{\lambda,e}$ if $J \in E^-_{\mu,e}$, and $J' \in E^-_{\lambda,e}$ if $J \in E^+_{\mu,e}$.

- **Step 10.** The elements of $B_{\lambda,d}$, $E^+_{\lambda,e} \sqcup E^-_{\lambda,e}$ and C_λ are determined by Step 8 and 9. Therefore one can define

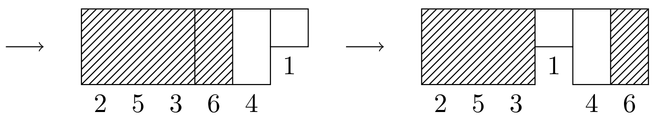
$$\begin{aligned} &I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \\ &\in \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda)\right) \sqcup \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right). \end{aligned}$$

  Since this algorithm does not change the depth of the painted columns, one can see $I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}_{n,d,e}$.

We illustrate this algorithm with an example $(2, 1, 2, 2, 2) \in B_{(2,1,1,1,1),11}$, $\{1, 3, 5\} \in E^-_{(2,1,1,1,1),4}$, $(2, 5)(1)(3)(4)(6) \in C_{(2,1,1,1,1)}$ where $n = 6$, $d = 11$, $e = 4$. One obtains the following left diagram after Step 3, and the right one after Step 4.



In this case, $\delta_0 = 2$ and $\rho = (2, 5)(3)(6)$, $i_1 = 2$, $i_2 = 3$. Since $(i_1, i_2)\rho = (2, 3)(2, 5)(3)(6) = (2, 5, 3)(6)$, one obtains the following left diagram after Step 7 and the right one after Step 8.



Therefore we have

$$\begin{aligned} &I\big((2, 1, 2, 2, 2), \{1, 3, 5\}, (2, 5)(1)(3)(4)(6)\big) \\ =&\big((2, 1, 2, 2), \{1, 4\}, (2, 5, 3)(1)(4)(6)\big) \\ \in& B_{(3,1,1,1),11} \times E^+_{(3,1,1,1),4} \times C_{(3,1,1,1)}. \end{aligned}$$

Since this algorithm changes the parity of the number of painted blocks in the Step 7,

$$I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right) \setminus \mathrm{AD}_{n,d,e}$$

for $\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda)$, and

$$I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda)$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right) \setminus \mathrm{AD}_{n,d,e}$. Since I is involution, we obtain a bijection

$$f_{n,d,e,1} : \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \to \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right) \setminus \mathrm{AD}_{n,d,e}.$$

Next, we define

$$f_{n,d,e,2}\big((a_1, a_2, \ldots, a_n), J, \sigma\big) \in \mathrm{AD}_{n,d,e}$$

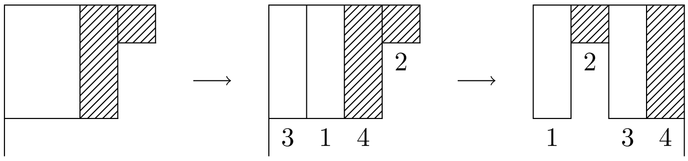
for d_1, d_2 satisfying $d_1 + d_2 = d$ and $(a_1, a_2, \ldots, a_n) \in A_{n,d_1}$, $J \in D_{n,d_2,e}$, and $\sigma \in \mathfrak{S}_n$ by the following algorithm:

- **Step 1.** Write $J = \{j_1, j_2, \ldots, j_e\}$ with $0 \le j_1 < j_2 < \cdots < j_e \le n-1$, and draw the rim of the Young diagram of the partition of d obtained by adding parts $j_1, j_2, \ldots, j_e$ to $(1^{a_1} 2^{a_2} \cdots n^{a_n})$. If $j_1 = 0$, we consider the diagram has a row of width 0 at the bottom.
- **Step 2.** Split the diagram into blocks of columns by depth.
- **Step 3.** For $1 \le m \le e$, the diagram has a row of width j_m by Step 1. Write the highest such row the i_mth row.
- **Step 4.** For $1 \le m \le e$, the diagram has a column of depth $i_m - 1$ by Step 3. Split the most right such column and paint it. This step makes e painted columns whose depths are all distinct.
- **Step 5.** Use Step 2 to Step 6 of the algorithm for $f_{n,d}$ in Theorem 1.1 for the diagram, and take the corresponding $\lambda \vdash n$ and elements of $B_{\lambda,d}$ and C_λ.
- **Step 6.** Put J' be the subset of $\{1, 2, \ldots, l(\lambda)\}$ such that $j \in J'$ if and only if the jth block from the left of the diagram is painted. One can see $J' \in E^+_{\lambda,e}$ since e is even.
- **Step 7.** The elements of $B_{\lambda,d}$, $E^+_{\lambda,e}$ and C_λ are determined by Step 5 and 6. Therefore one can define

  $$f_{n,d,e}\big((a_1, a_2, \ldots, a_n), J, \sigma\big) \in \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda).$$

  Moreover, this is an element of $\mathrm{AD}_{n,d,e}$ because the depths of the painted columns in the diagram are all distinct.

We illustrate this algorithm with an example $(0,0,1,1) \in A_{4,7}$, $\{0,3\} \in D_{4,3,2}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \in \mathfrak{S}_4$, where $n = 4$, $d = 10$, and $e = 2$. In this case $j_1 = 0$ and $j_2 = 3$, thus we draw in Step 1 the rim of the Young diagram of $(4,3,3,0) \vdash 10$, which is obtained by adding parts $0, 3$ to $\big(1^0 2^0 3^1 4^1\big) = (4,3)$. Since $i_1 = 4$ and $i_2 = 2$, we get the following left diagram after Step 4. We use Step 2 to Step 6 of the algorithm for Theorem 1.1 in Step 5, and we get the following middle diagram after the Step 3 and the right one after the Step 4.



Thus we obtain

$$\begin{aligned}&f_{4,10,2}\left((0,0,1,1),\{0,3\},\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}\right)\\=&\big((3,1,3,3),\{2,4\},(1)(2)(3)(4)\big)\\\in&B_{(1,1,1,1),10}\times E^{+}_{(1,1,1,1),2}\times C_{(1,1,1,1)}.\end{aligned}$$

We can construct the inverse function

$${f_{n,d,e,2}}^{-1}:\mathrm{AD}_{n,d,e}\to\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,e}\times\mathfrak{S}_n)$$

by defining

$${f_{n,d,e,2}}^{-1}\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)\in\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,e}\times\mathfrak{S}_n)$$

for $\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)\in\mathrm{AD}_{n,d,e}$ by the following algorithm:

- **Step 1.** Use Step 1 and 2 of the algorithm for ${f_{n,d}}^{-1}$ in Theorem 1.1, and obtain a diagram of blocks. The ith block from the left is of width μ_i and depth b_i, and each column has number defined by τ.
- **Step 2.** For $j\in J'$, paint the jth block from the left. There are e painted columns of all distinct depths.
- **Step 3.** Arrange the blocks by the following rules:
  1. Put deeper one to the left.
  2. If there are blocks of the same depth, put painted one to the right.
  3. If there are blocks of the same depth and painting, put one which has the smallest number to the right.
- **Step 4.** Let the depths of the painted columns are $i_1,i_2,\dots,i_e$ with the condition $i_1>i_2>\cdots>i_e$.
- **Step 5.** For $1\le m\le e$, write the (i_m+1)th row has width $j_m\ge 0$.
- **Step 6.** Define

  $${f_{n,d,e,2}}^{-1}\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)=\big((a_1,a_2,\dots,a_n),J,\sigma\big)$$

  by the following condition:
  1. $J=\{j_1,j_2,\dots,j_e\}$.
  2. $(1^{a_1}2^{a_2}\cdots n^{a_n})$ is the partition obtained by removing the parts $j_1,j_2,\dots,j_e$ from the shape of the diagram.

3. Each column of the diagram has the numbers

$$\sigma(1), \sigma(2), \ldots, \sigma(n)$$

from the left.

Therefore we have defined a bijection

$$f_{n,d,e,2} : \bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,e} \times \mathfrak{S}_n) \to \mathrm{AD}_{n,d,e}.$$

By two bijections $f_{n,d,e,1}$ and $f_{n,d,e,2}$ we have constructed, now we have a bijection

$$\begin{aligned} f_{n,d,e} : &\left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,e} \times \mathfrak{S}_n) \right) \\ &\to \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \end{aligned}$$

for $d \in \mathbb{N}$ and even number e satisfying $0 \le e \le n$. In a similar way, one can construct a bijection

$$\begin{aligned} f_{n,d,e} : &\left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,e} \times \mathfrak{S}_n) \right) \\ &\to \bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \end{aligned}$$

for $d \in \mathbb{N}$ and odd number e satisfying $0 \le e \le n$. By these bijections, we complete the proof of Theorem 2.1.

Next, we prove Theorem 2.2. One obtains

$$\begin{aligned} n! \prod_{i=1}^n \frac{t^{i-1} - q}{1 - t^i} =& n! \left(\prod_{i=1}^n \frac{1}{1-t^i} \right) \left(\prod_{i=1}^n (t^{i-1} - q) \right) \\ =& |\mathfrak{S}_n| \left(\sum_{d_1=0}^\infty |A_{n,d_1}| t^{d_1} \right) \left(\sum_{d_2=0}^\infty \sum_{e=0}^n |D_{n,d_2,n-e}| t^{d_2} (-q)^e \right) \\ =& \sum_{d=0}^\infty \sum_{e=0}^n \left((-1)^e \sum_{d_1+d_2=d} |A_{n,d_1} \times D_{n,d_2,n-e} \times \mathfrak{S}_n| \right) t^d q^e \end{aligned}$$

and

$$\begin{aligned} & n! \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \\ =& \sum_{\lambda \vdash n} \epsilon_\lambda \cdot \frac{n!}{z_\lambda} \left(\prod_{i=1}^{l(\lambda)} \frac{1}{1 - t^{\lambda_i}} \right) \left(\prod_{i=1}^{l(\lambda)} (1 - q^{\lambda_i}) \right) \\ =& \sum_{\lambda \vdash n} \epsilon_\lambda |C_\lambda| \left(\sum_{d=0}^\infty |B_{\lambda,d}| t^d \right) \left(\sum_{e=0}^n \left(\left| E^+_{\lambda,e} \right| - \left| E^-_{\lambda,e} \right| \right) q^e \right) \end{aligned}$$

$$=\sum_{d=0}^{\infty}\sum_{e=0}^{n}\left(\sum_{\lambda\vdash n}\left(\epsilon_\lambda\left|B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda\right|-\epsilon_\lambda\left|B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda\right|\right)\right)t^d q^e$$

$$=\sum_{d=0}^{\infty}\sum_{e=0}^{n}\left(\sum_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}\left|B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda\right|-\sum_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}\left|B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda\right|\right.$$

$$\left.-\sum_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}\left|B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda\right|+\sum_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}\left|B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda\right|\right)t^d q^e.$$

Therefore it suffices to construct a bijection

$$g_{n,d,e}:\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\right)$$
$$\sqcup\left(\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,n-e}\times\mathfrak{S}_n)\right)$$
$$\to\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\right)$$

for $d\in\mathbb{N}$ and even number e satisfying $0\le e\le n$, and

$$g_{n,d,e}:\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\right)$$
$$\sqcup\left(\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,n-e}\times\mathfrak{S}_n)\right)$$
$$\to\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\right)\sqcup\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\right)$$

for $d\in\mathbb{N}$ and odd number e satisfying $0\le e\le n$. Here we construct a bijection $g_{n,d,e}$ for $d\in\mathbb{N}$ and even number e satisfying $0\le e\le n$. We construct $g_{n,d,e}$ by constructing three bijections

$$g_{n,d,e,1}:\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\to\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{-}_{\lambda,e}\times C_\lambda)\right)\setminus\mathrm{AD}'_{n,d,e},$$

$$g_{n,d,e,2}:\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\to\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}}(B_{\lambda,d}\times E^{+}_{\lambda,e}\times C_\lambda)\right)\setminus\mathrm{AD}'_{n,d,e},$$

$$g_{n,d,e,3}:\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,n-e}\times\mathfrak{S}_n)\to\mathrm{AD}'_{n,d,e},$$

where $\mathrm{AD}'_{n,d,e}$ is the set of all elements

$$
\begin{aligned}
&\big((b_1, b_2, \ldots, b_{l(\lambda)}), J, \tau\big) \\
&\in \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right)
\end{aligned}
$$

such that $\lambda_j = 1$ for all $j \in \{1, 2, \ldots, l(\lambda)\} \setminus J$, and b_j for $j \in \{1, 2, \ldots, l(\lambda)\} \setminus J$ are all distinct.

We construct $g_{n,d,e,1}$ and $g_{n,d,e,2}$ by the involution I on

$$
\left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}
$$

defined by the following algorithm:

- **Step 1.** Take $\mu \vdash n$ and

$$
\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in (B_{\mu,d} \times E^-_{\mu,e} \times C_\mu) \sqcup (B_{\mu,d} \times E^+_{\mu,e} \times C_\mu)
$$

  satisfying $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}'_{n,d,e}$.

- **Step 2.** Use Step 1 and 2 of the algorithm for ${f_{n,d}}^{-1}$ in Theorem 1.1, and obtain a diagram of blocks. The ith block from the left is of width μ_i and depth b_i, and each column has number defined by τ.

- **Step 3.** For $j \in J$, paint the jth block from the left. There are e painted columns because $J \in E^+_{\mu,e} \sqcup E^-_{\mu,e}$.

- **Step 4.** Rearrange the blocks by the following rules:

  1. Put deeper one to the left.
  2. If there are blocks of the same depth, put painted one to the left.
  3. If there are blocks of the same depth and painting, put one which has the smallest number to the left.

- **Step 5.** Since $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}'_{n,d,e}$, there is an unpainted block with width at least 2, or there are unpainted blocks of width 1 and same depth. Therefore there is a depth δ such that there are at least two unpainted columns of depth δ. Let δ_0 be the deepest such depth δ.

- **Step 6.** Write the unpainted blocks of depth δ_0 $X_1, X_2, \ldots$ from left. For each j, write the numbers written on each column of X_j $q_{j,1}, q_{j,2}, \ldots$ from left. Let

$$
\rho = (q_{1,1}, q_{1,2}, \ldots)(q_{2,1}, q_{2,2}, \ldots) \cdots \in \mathfrak{S}_n,
$$

  and i_1, i_2 be the smallest two numbers in

$$
\{q_{1,1}, q_{1,2}, \ldots\} \cup \{q_{2,1}, q_{2,2}, \ldots\} \cup \cdots
$$

  which satisfies $i_1 < i_2$.

- **Step 7.** $(i_1, i_2)\rho$ is written $(r_{1,1}, r_{1,2}, \dots)(r_{2,1}, r_{2,2}, \dots)\cdots$ by some disjoint cycles $(r_{j,1}, r_{j,2}, \dots)$ satisfying $r_{j,1} = \min\{r_{j,1}, r_{j,2}, \dots\}$ for each j, and $r_{1,1} < r_{2,1} < \cdots$. For each j, let Y_j be the unpainted block whose width is the length of $(r_{j,1}, r_{j,2}, \dots)$ and whose depth is δ_0, and whose each column has the numbers $r_{j,1}, r_{j,2}, \dots$ from the left. Arrange $Y_1, Y_2, \dots$ from the left, and exchange the all unpainted blocks of depth δ_0 of the diagram made in Step 4 for them.

  The operation of (i_1, i_2) on ρ changes the sign of ρ. Hence $(i_1, i_2)\rho$ consists of even numbers of cycles if ρ consists of odd numbers of them, and vice versa. Therefore this step exchanges the parity of the number of unpainted blocks in the diagram.

- **Step 8.** Use Step 4 to Step 6 of the algorithm for $f_{n,d}$ in Theorem 1.1, and take the corresponding partition $\lambda \vdash n$ and the elements of $B_{\lambda,d}$ and C_λ.

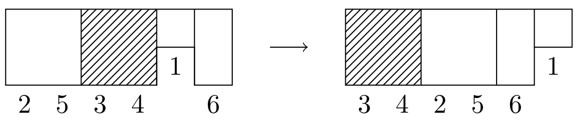
- **Step 9.** Put J' be the subset of $\{1, 2, \dots, l(\lambda)\}$ such that $j \in J'$ if and only if the jth block from the left of the diagram is painted. One can see $J' \in E^+_{\lambda,e}$ if $J \in E^+_{\mu,e}$, and $J' \in E^-_{\lambda,e}$ if $J \in E^-_{\mu,e}$.

- **Step 10.** The elements of $B_{\lambda,d}$, $E^+_{\lambda,e} \sqcup E^-_{\lambda,e}$ and C_λ are determined by Step 8 and 9. Therefore one can define

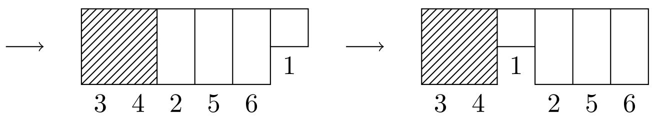
$$\begin{aligned}
&I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \\
&\in \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda)\right) \sqcup \left(\bigsqcup_{\lambda \vdash n} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right).
\end{aligned}$$

  Since this algorithm does not change the depth of the unpainted columns, one can see $I\big((b_1, b_2, \dots, b_{l(\mu)}), J, \tau\big) \notin \mathrm{AD}'_{n,d,e}$.

We illustrate this algorithm with an example $(2, 2, 1, 2) \in B_{(2,2,1,1),11}$, $\{2\} \in E^-_{(2,2,1,1),2}$, $(2, 5)(3, 4)(1)(6) \in C_{(2,2,1,1)}$ where $n = 6$, $d = 11$, $e = 2$. One obtains the following left diagram after Step 3, and the right one after Step 4.



In this case, $\delta_0 = 2$ and $\rho = (2, 5)(6)$, $i_1 = 2$, $i_2 = 5$. Since $(i_1, i_2)\rho = (2, 5)(2, 5)(6) = (2)(5)(6)$, one obtains the following left diagram after Step 7 and the right one after Step 8.



Therefore we have

$$\begin{aligned}
&g_{6,11,2}\big((2, 2, 1, 2), \{2\}, (2, 5)(3, 4)(1)(6)\big) \\
=&\big((2, 1, 2, 2, 2), \{1\}, (3, 4)(1)(2)(5)(6)\big) \\
\in& B_{(2,1,1,1,1),11} \times E^-_{(2,1,1,1,1),2} \times C_{(2,1,1,1,1)}.
\end{aligned}$$

Since this algorithm changes the parity of the number of unpainted blocks in the Step 7,

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup\limits_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda)$,

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda)$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup\limits_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}$,

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda)$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup\limits_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}$, and

$$I\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}$$

for $\big((b_1, b_2, \ldots, b_{l(\mu)}), J, \tau\big) \in \bigsqcup\limits_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda)$. Since I is involution, we obtain bijections

$$g_{n,d,e,1} : \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda) \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{-}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e},$$

$$g_{n,d,e,2} : \bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda) \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^{+}_{\lambda,e} \times C_\lambda) \right) \setminus \mathrm{AD}'_{n,d,e}.$$

Next, we define

$$g_{n,d,e,3}\big((a_1, a_2, \ldots, a_n), J, \sigma\big) \in \mathrm{AD}'_{n,d,e}$$

for d_1, d_2 satisfying $d_1 + d_2 = d$ and $(a_1, a_2, \ldots, a_n) \in A_{n,d_1}$, $J \in D_{n,d_2,n-e}$ and $\sigma \in \mathfrak{S}_n$ by the following algorithm:

- **Step 1.** Write $J = \{j_1, j_2, \ldots, j_{n-e}\}$ with $0 \le j_1 < j_2 < \cdots < j_{n-e} \le n-1$, and draw the rim of the Young diagram of the partition of d obtained by adding parts $j_1, j_2, \ldots, j_e$ to $(1^{a_1}2^{a_2}\cdots n^{a_n})$. If $j_1 = 0$, we consider the diagram has a row of width 0 at the bottom.

- **Step 2.** Split the diagram into blocks of columns by depth.

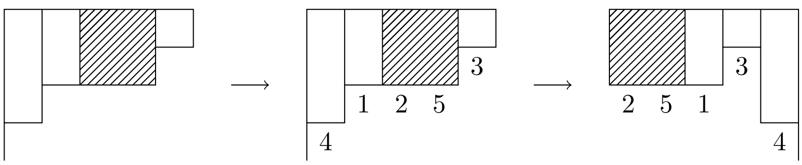
- **Step 3.** Split the (j_m+1)th column from the left for $1 \le m \le n-e$, and paint the other e columns.

- **Step 4.** Use Step 2 to Step 6 of the algorithm for $f_{n,d}$ in Theorem 1.1 for the diagram, and take the corresponding $\lambda \vdash n$ and elements of $B_{\lambda,d}$ and C_λ.

- **Step 5.** Put J' be the subset of $\{1, 2, \ldots, l(\lambda)\}$ such that $j \in J'$ if and only if the jth block from the left of the diagram is painted. One can see $J' \in E^+_{\lambda,e} \sqcup E^-_{\lambda,e}$.

- **Step 6.** There are $|J'|$ painted blocks, $n-e$ unpainted blocks, e painted columns, and $n-e$ unpainted columns. Hence if λ is even, $n-l(\lambda) = e-|J'|$ is even and $|J'|$ is even, therefore $J' \in E^+_{\lambda,e}$. Similarly, $J' \in E^-_{\lambda,e}$ if λ is odd. Now we have the element of
$$\left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{even}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda)\right) \sqcup \left(\bigsqcup_{\substack{\lambda\vdash n\\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda)\right)$$
by Step 4, 5 and 6. Moreover, this is an element of $\mathrm{AD}'_{n,d,e}$ because the depths of the unpainted columns in the diagram are all distinct. Therefore one can define
$$g_{n,d,e}\big((a_1, a_2, \ldots, a_n), J, \sigma\big) \in \mathrm{AD}'_{n,d,e}.$$

We illustrate this algorithm with an example
$$(0,0,0,0,1) \in A_{5,5}, \quad \{0,1,4\} \in D_{5,5,3} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix} \in \mathfrak{S}_5,$$
where $n = 5$, $d = 10$, and $e = 2$. In this case $j_1 = 0$, $j_2 = 1$ and $j_3 = 4$, thus we draw in Step 1 the rim of the Young diagram of $(5,4,1,0) \vdash 10$, which is obtained by adding parts $0, 1, 4$ to $\big(1^0 2^0 3^0 4^0 5^1\big) = (5)$. Hence we get the following left diagram after Step 3. We use Step 2 to Step 6 of the algorithm for Theorem 1.1 in Step 4, and we get the following middle diagram after the Step 3 and the right one after the Step 4.



Thus we obtain

$$\begin{aligned}
&g_{5,10,3}\left((0,0,0,0,1),\{0,1,4\},\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & 2 & 5 & 3\end{pmatrix}\right)\\
=&\big((2,2,1,3),\{1\},(2,5)(1)(3)(4)\big)\\
\in&B_{(2,1,1,1),10}\times E^{-}_{(2,1,1,1),2}\times C_{(2,1,1,1)}.
\end{aligned}$$

We can construct the inverse function

$$g_{n,d,e,3}{}^{-1}:\mathrm{AD}'_{n,d,e}\to\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,n-e}\times\mathfrak{S}_n).$$

by defining

$$g_{n,d,e,3}{}^{-1}\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)\in\bigsqcup_{d_1+d_2=d}(A_{n,d_1}\times D_{n,d_2,n-e}\times\mathfrak{S}_n)$$

for $\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)\in\mathrm{AD}'_{n,d,e}$ by the following algorithm:

- **Step 1.** Use Step 1 and 2 of the algorithm for $f_{n,d}{}^{-1}$ in Theorem 1.1, and obtain a diagram of blocks. The ith block from the left is of width μ_i and depth b_i, and each column has number defined by τ.
- **Step 2.** For $j\in J'$, paint the jth block from the left. There are $n-e$ unpainted columns of all distinct depths.
- **Step 3.** Arrange the blocks by the following rules:
  1. Put deeper one to the left.
  2. If there are blocks of the same depth, put painted one to the right.
  3. If there are blocks of the same depth and painting, put one which has the smallest number to the right.
- **Step 4.** Let the depths of the unpainted columns are $i_1,i_2,\dots,i_{n-e}$ with the condition $i_1>i_2>\cdots>i_{n-e}$.
- **Step 5.** For $1\le m\le n-e$, write the (i_m+1)th row has width $j_m\ge 0$.
- **Step 6.** Define

  $$g_{n,d,e,3}{}^{-1}\big((b_1,b_2,\dots,b_{l(\mu)}),J',\tau\big)=\big((a_1,a_2,\dots,a_n),J,\sigma\big)$$

  by the following condition:
  1. $J=\{j_1,j_2,\dots,j_{n-e}\}$.
  2. $(1^{a_1}2^{a_2}\cdots n^{a_n})$ is the partition obtained by removing the parts $j_1,j_2,\dots,j_{n-e}$ from the shape of the diagram.
  3. Each column of the diagram has the numbers

     $$\sigma(1),\sigma(2),\dots,\sigma(n)$$

     from the left.

Therefore we have defined a bijection

$$g_{n,d,e,3} : \bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,n-e} \times \mathfrak{S}_n) \to \mathrm{AD}'_{n,d,e},$$

By three bijections $g_{n,d,e,1}$, $g_{n,d,e,2}$ and $g_{n,d,e,3}$ we have constructed, now we have a bijection

$$\begin{aligned} g_{n,d,e} : & \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \\ & \sqcup \left(\bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,n-e} \times \mathfrak{S}_n) \right) \\ & \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \end{aligned}$$

for $d \in \mathbb{N}$ and even number e satisfying $0 \leq e \leq n$. In a similar way, one can construct a bijection

$$\begin{aligned} g_{n,d,e} : & \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \\ & \sqcup \left(\bigsqcup_{d_1+d_2=d} (A_{n,d_1} \times D_{n,d_2,n-e} \times \mathfrak{S}_n) \right) \\ & \to \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{even}}} (B_{\lambda,d} \times E^-_{\lambda,e} \times C_\lambda) \right) \sqcup \left(\bigsqcup_{\substack{\lambda \vdash n \\ \lambda:\text{odd}}} (B_{\lambda,d} \times E^+_{\lambda,e} \times C_\lambda) \right) \end{aligned}$$

for $d \in \mathbb{N}$ and odd number e satisfying $0 \leq e \leq n$. By these bijections, we complete the proof of Theorem 2.2.

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